THEORIZING THE MATHEMATICAL POINT OF BUILDING ON STUDENT MATHEMATICAL THINKING

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Despite the fact that the mathematics education field recognizes the critical role that student thinking plays in high quality instruction, little is known about productive use of the in-the-moment student thinking that emerges in the context of whole-class discussion. We draw on and extend the work of others to theorize the mathematical understanding an instance of such student thinking can be used to build towards—the mathematical point (MP). An MP is a mathematical statement of what could be gained from considering a particular instance of student thinking. Examples and non-examples are used to illustrate nuances in the MP construct. Articulating the MP for an instance of student thinking is requisite for determining whether there is instructional value in pursuing that thinking.

The field of mathematics education recognizes the critical role student mathematical thinking plays in planning and implementing quality mathematics instruction (e.g., National Council of Teachers of Mathematics [NCTM], 2014). Researchers have made progress on understanding how instruction can be improved by using tasks that are likely to engage students in meaningful mathematical activity and by working to maintain the cognitive demand of those tasks throughout instruction (e.g., Stein & Lane, 1996). We also know many of the benefits of teachers understanding common ways that students think about and develop mathematical ideas (e.g., Fennema et al., 1996). We know less, however, about productive ways of taking advantage of the student mathematical thinking that emerges during instruction. Recent work (e.g., Smith & Stein, 2011) has begun to help us understand how to effectively use written records of student work, but much less is known about how to effectively use the in-the-moment mathematical thinking that emerges during classroom mathematics discourse. We need to understand this important aspect of effective use of student thinking because whole-class discussion is fertile ground for the emergence of valuable student mathematical thinking (Van Zoest et al., 2015a, 2015b), yet many teachers, especially novices, fail to notice or to act on opportunities to use this valuable thinking to further mathematical understanding (Peterson & Leatham, 2009; Stockero, Van Zoest, & Taylor, 2010).

Our work investigating teachers' use of in-the-moment instances of high potential student thinking to further students' mathematical understanding during whole-class discussion has led us to conclude that an important reason for the slow pace of reform in this area is that what exactly can be learned from making a particular instance of

2016. In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 4, pp. 323–330. Szeged, Hungary: PME.

student thinking the object of discussion has been under theorized. Thus, the purpose of this paper is to theorize the mathematical point that an instance of student thinking can be used to build towards. Before beginning that theorizing, we first outline the theoretical framework that guides our thinking and then situate our thinking in the context of other related research.

THEORETICAL FRAMEWORK

The MOST research group (e.g., Leatham, Peterson, Stockero, & Van Zoest, 2015; Van Zoest, Leatham, Peterson, & Stockero, 2013) defined MOSTs—**M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking—as "instances of student thinking that have considerable potential at a given moment to become the object of rich discussion about important mathematical ideas" (Leatham et al., 2015, p. 90). They conceptualized MOSTs as occurring in the intersection of three critical characteristics of classroom instances: student mathematical thinking, significant mathematics, and pedagogical opportunities. For each characteristic, two criteria were provided to determine whether an instance of student thinking embodies that characteristic. Foundational to our work is the student mathematical thinking characteristic, for which the two criteria are student mathematics and mathematical point. To meet the student mathematics criterion, one must have sufficient evidence to make a reasonable inference about the mathematical thinking a student is expressing. The articulation of this mathematical thinking is called the *student mathematics* (SM) of the instance. To meet the mathematical point criterion, one must be able to "articulate a mathematical idea that is closely related to the student mathematics of the instance—what we call a *mathematical point*" (p. 92). It is this use of *mathematical point (MP)* that we theorize in this paper.

MOSTs are instances of student thinking worth *building* on—that is, "student thinking worth making the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea" (Van Zoest et al., 2015b, p. 4). Such use encapsulates core ideas of current thinking about effective teaching and learning of mathematics, including social construction of knowledge and the importance of mathematical discourse (NCTM, 2014). Thus, building on MOSTs is a particularly productive way for teachers to engage students in meaningful mathematical learning. In this paper, we both draw on and extend the MOST framework by theorizing the MP—the mathematical understanding particular instances of student thinking can be used to build towards.

RELATED RESEARCH

Perhaps the work most closely related to this theorizing, at least on the surface, is Laurie Sleep's 2009 dissertation, *Teaching to the Mathematical Point: Knowing and Using Mathematics in Teaching*. Sleep, however, defines mathematical point "to include the mathematical learning goals for an activity, as well as the connection between the activity and its goals" (p. 13). This is a broad definition that foregrounds the meaning of "point" as "something that is the focus of attention, consideration, or purpose" and backgrounds the meaning of point as "a separate, or single item, article, or element in an extended whole" (Oxford English Dictionary [OED] cited in Sleep, 2009, p. 13). The first column of Figure 1 lists mathematical points articulated in Sleep's dissertation. What is notable about these points is the lack of specifics they provide about the mathematical ideas related to them. Consider, for example, the first point in Figure 1. Although "reviewing and practicing strategies for adding multiple addends" (Sleep, 2009, p. 107) is important to be doing in a 2nd grade class, the statement does not say anything about the mathematics that makes up those strategies. That is, it fails to articulate the mathematical idea that is to be learned.

Mathematical Points from Sleep (2009)	*Mathematical Understandings from Charles (2005)*
"reviewing and practicing strategies for adding multiple addends" (p. 107)	"Numbers can be broken apart and grouped in different ways to make calculations simpler." (p. 16)
"learning that halves are two equal parts" (p. 162)	"The bottom number in a fraction tells how many equal parts the whole or unit is divided into. The top number tells how many equal parts are indicated." (p. 13)
"teaching the addition and subtraction algorithm [for fractions]" (p. 244)	"Fractions with unlike denominators are renamed as equivalent fractions with like denominators to add and subtract." (p. 16)

Figure 1: Comparison of Sleep's (2009) Mathematical Points and Charles' (2005) Mathematical Understandings.

Although the importance of teachers having mathematical goals in mind for their teaching has been well established (e.g., Corey et al., 2010; NCTM 2014), most research, like Sleep's (2009), has remained at the level of looking at whether teachers have mathematical goals and how those goals affect their instruction (e.g., Fernandez, Cannon, & Chokshi, 2003), rather than investigating the articulation of the goals and how that articulation affects teachers' ability to support their students' learning. Some work has been done around the articulation of intended instructional outcomes, however, in the context of courses on pedagogy. For example, the Brigham Young University Mathematics Educators (n.d., unpublished manuscript) developed a document for supporting preservice secondary school teachers in writing lesson goals focused on mathematical concepts that is now used by several universities. They emphasized that a *key concept* is not a topic or a step-by-step method for doing something; rather, it is "something that you want your students to *understand*. Concepts deal with *meaning*, *why* something works, *ways of imagining* or *seeing things*, and *connections*" (p. 1, italics in original).

Charles (2005), to "initiate a conversation about the notion of Big Ideas in mathematics" (p. 9), proposed a set of Big Ideas for elementary and middle school and their corresponding *mathematical understandings*. Charles described a mathematical understanding as "an important idea students need to learn because it contributes to understanding the Big Idea" (p. 10). The second column of Figure 1 lists mathematical understandings from Charles (2005) that bear some relationship to Sleep's (2009) mathematical points. Charles' mathematical understandings do articulate the mathematical idea that is to be learned, and thus they are at a grainsize more appropriate for looking at the mathematical understanding particular instances of student thinking can be used to build towards.

DEFINING MATHEMATICAL POINT (MP)

Although Charles (2005) did not formally define mathematical understanding in his paper, drawing on his examples and explanations, we use the term *mathematical understanding* to refer to *a concise statement of a non-subjective truth about mathematics*. This definition specifies something that students can actually come to understand, as opposed to a topic for them to study or an outcome of their learning. Articulating mathematical understandings is useful for a number of teaching-related activities, such as determining goals of a lesson, analysing the mathematics students might learn from a task, and guiding the formulation of questions to ask in the midst of a lesson. Yet another reason for articulating mathematical understandings, and the one that is the focus of this paper, is to determine whether student thinking is worth building on in the moment in which it occurs.

Our focus is on instances of student mathematical thinking that emerge during whole-class discussion. We follow Leatham et al. (2015) in defining an *instance* as "an observable student action or small collection of connected actions (such as a verbal expression combined with a gesture)" (p. 92). In our ongoing research, we have found that for instances of student thinking for which *student mathematics (SM)* can be inferred, one can articulate related *mathematical understanding(s)*. Since student thinking is not always constrained by the teachers' plan for the lesson, these mathematical understandings may or may not be within the confines of the planned lesson. Additionally, the mathematical understandings that are within the confines of the planned lesson may or may not be most closely related to the SM. Although we agree with Hintz and Kazemi (2014) that it is important that "the discussion goal acts as a compass as teachers navigate classroom talk" (p. 37), we also contend that a parallel goal is to honor student thinking. That is, making a decision about whether or not to pursue a particular instance of student mathematical thinking requires first identifying the SM of the thinking and then identifying the mathematical understanding most closely related to it. Otherwise, there is a risk of undermining a core principle of quality mathematics instruction—that of positioning students as legitimate mathematical thinkers (e.g., NCTM, 2014).

Thus, in the context of our work on productive use of student mathematical thinking during instruction, a *mathematical point (MP)* is the *mathematical understanding* that (1) students could gain from considering a particular instance of student thinking and (2) is most closely related to the SM of the thinking. That is, the MP is a mathematical statement of what could be gained as a result of students making sense of the mathematics contained in the expression of the student thinking. Note that it is only when the MP is articulated that a clear decision can be made about whether the student thinking should be pursued. (Leatham et al., 2015, describe a tool for distinguishing instances of student thinking that provide opportunities to productively build on students' mathematical thinking from those that do not—the MOST Analytic Framework.)

We have identified four things to keep in mind when considering MPs. First, an MP exists only in relation to a specific instance of student thinking. That is, unlike a mathematical understanding, which can stand alone, an MP cannot. Specifically, one must talk about an MP in relation to what can be gained from considering a particular instance of student thinking. Second, in order to be an MP, the mathematical understanding must be gained from considering the student thinking itself. An instance of student thinking may often prompt teachers to ask a question, introduce an idea or pose a task that furthers student learning of a mathematical understanding related to the instance. Although these are important teaching tasks that *use* student thinking, we want to be clear that we do not consider them *building* on student thinking. In order for building to occur, the thinking itself must become the object of discussion. Third, not all instances of student thinking give rise to an MP. For example, suppose a student asked, "What is the formula for the volume of a cube?" This instance of mathematical thinking is related to the mathematical understanding: *The formula for the volume of a cube with side s is* s^3. That mathematical understanding, however, is not something that students could gain from considering this particular instance of student thinking. They might be able to recall it, or they might be able to figure it out from a task that the teacher poses in response to the instance, but it would not result from considering the student thinking. Fourth, there are acceptable variations in the articulation of SMs, mathematical understandings, and MPs. What is presented here is the consensus of the authors, but other articulations may also be defendable.

To further illustrate our theorizing, Figure 2 contains instances of student mathematical thinking, the MP that would serve as the discussion goal of the conversation in which the instance of student thinking is the object of discussion, an example of a mathematical understanding that does not meet the "most closely related" criteria for that thinking and an example of a related statement that is not a mathematical understanding. Recall that MPs are a subset of mathematical understandings, thus both the second and third column contain examples of mathematical understandings.

Instances of Student Mathematical Thinking	Mathematical Point	Not the Most Closely Related Mathematical Understanding	Not a Mathematical Understanding
1. In an introductory lesson on adding fractions with like denominators, a student writes the following on the board: 2/5 + 1/5 = 3/10.	Adding fractional pieces of the same size changes the number of pieces, but not the size of the pieces.	Adding two quantities means combining the amounts together.	How to get a common denominator when adding fractions.
2. During the second day of a lesson on solving simple linear equations, when the teacher solves the equation $x + 2 = 5$ and writes $x = 3$ on the board, a student remarks, "Hey, wait a minute, yesterday you said x equals two and today you're saying x equals three!"	A letter can be used to represent an unknown quantity in an equation and can represent different quantities for different equations.	"Letters are used in mathematics to represent generalized properties, unknowns in equations, and relationships between quantities." (Charles, 2005, p. 18)	The meaning of variable.
3. In a beginning algebra lesson on solving simple linear equations, a student says, "To get m alone on the left side of the equation $m – 12 = 5$, you can subtract 12."	Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation.	Adding a number and subtracting that same number are inverse operations.	Solving linear equations.

Figure 2: Examples and Non-examples of Mathematical Points for Instances of Student Mathematical Thinking

Since the MP is the most closely related mathematical understanding, we first look at ways in which statements may fall short of being a mathematical understanding (see Column 4 of Figure 2). "How to get a common denominator when adding fractions," for example, states a mathematical process without explicating that process. "The meaning of variable," refers to a concept without elaborating what it is, while "Solving linear equations," merely states a topic. Note that all of these mathematical statements fail to specify the *non-subjective truth about mathematics* that the statement encapsulates.

The mathematical understandings in Column 3 of Figure 2 are mathematical understandings for the corresponding instances of student thinking, but they are not as closely related as those in the *Mathematical Point* column. For example, although closely related on the surface—Instance 1 is certainly about addition of two quantities—the MP for this instance better captures the specific non-subjective truth about mathematics that students could gain by making this instance of thinking the object of discussion. The importance of the MP being the mathematical understanding most closely related to the SM of the instance is related to the idea of honoring student thinking. For example, if the teacher were to turn the student thinking in Instance 2 over to the class and navigate the discussion (Hintz & Franke, 2014) toward the goal of better understanding how letters are used to represent unknowns in equations, the student likely would not feel that their thinking was the object of the discussion. Again, that is not to say that making the student thinking the object of discussion is always the optimal teaching move; rather, it is to say that articulating the MP allows teachers to make an informed decision about how best to respond to the thinking. If there is an MP, the MOST Analytic Framework (Leatham et al., 2015) is a mechanism for determining whether to make the thinking the object of discussion for the class or to address it in some other way.

CONCLUSION

An important reason that instruction based on student thinking has not lived up to its potential may be that our target has been too broad. Focusing on teachers' goals for the lesson lends itself to the teacher using student thinking to make the point the teacher has in mind, rather than building on student thinking. Changing the grainsize to the MP for the SM in instances of student thinking may be a productive shift in how we think about using student thinking as part of instruction that will allow us to achieve the full potential of instruction based on student thinking.

Acknowledgements

This research report is based on work supported by the U.S. National Science Foundation (NSF) under Grant Nos. 1220141, 1220357, and 1220148. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF. The authors thank Mary Ochieng and Elizabeth Fraser for their contributions to the ideas in this paper.

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