What's the Point?
Identifying Important Mathematics Underlying Student Thinking

Laura R. Van Zoest

Overview

- Context & Motivation for the Work
- Launch: An Instance of Student Thinking
- Definition of Mathematical Point
- Activity 1: Mathematical Point Card Sort
- Activity 2: Mathematical Points in Student Work
- Observations and next steps
- Connections to other important ideas

Context

Identifying MOSTs – *Mathematical Opportunities in Student Thinking* – high potential instances of student thinking

Supported by
Leveraging MOSTs: Developing a Theory of Productive Use of Student Mathematical Thinking
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Principal Investigators
Keith R. Leatham – Brigham Young University
Blake E. Peterson – Brigham Young University
Shari L. Stockero – Michigan Technological University
Laura R. Van Zoest – Western Michigan University

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Leveraging M … **Productive Use of Student**

a 4-yea … Science Foundation
-1220148)

and…
Napthalin Atanga (WMU)
Mary Ochieng (WMU)
Lindsay Merrill (BYU)
Elizabeth Fraser (WMU)
Annick Rougee (U of M)
Rachel Gunn (BYU)

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MOST Characteristics

Student Mathematical Thinking

Student Mathematics

Can the student mathematics be inferred?

Mathematical Point

Is there a mathematical point closely related to the student mathematics?

Mathematically Significant

Appropriate Mathematics

Is the mathematical point accessible to students with this level of mathematical experience, but not like to be already understood?

Central Mathematics

Is understanding the mathematical point a central goal for student learning in this classroom?

Pedagogical Opportunity

Opening

Does the expression of the student mathematics seem to create an intellectual need that, if met, will contribute to understanding the mathematical point of the instance?

Timing

Is now the right time to take advantage of the opening?

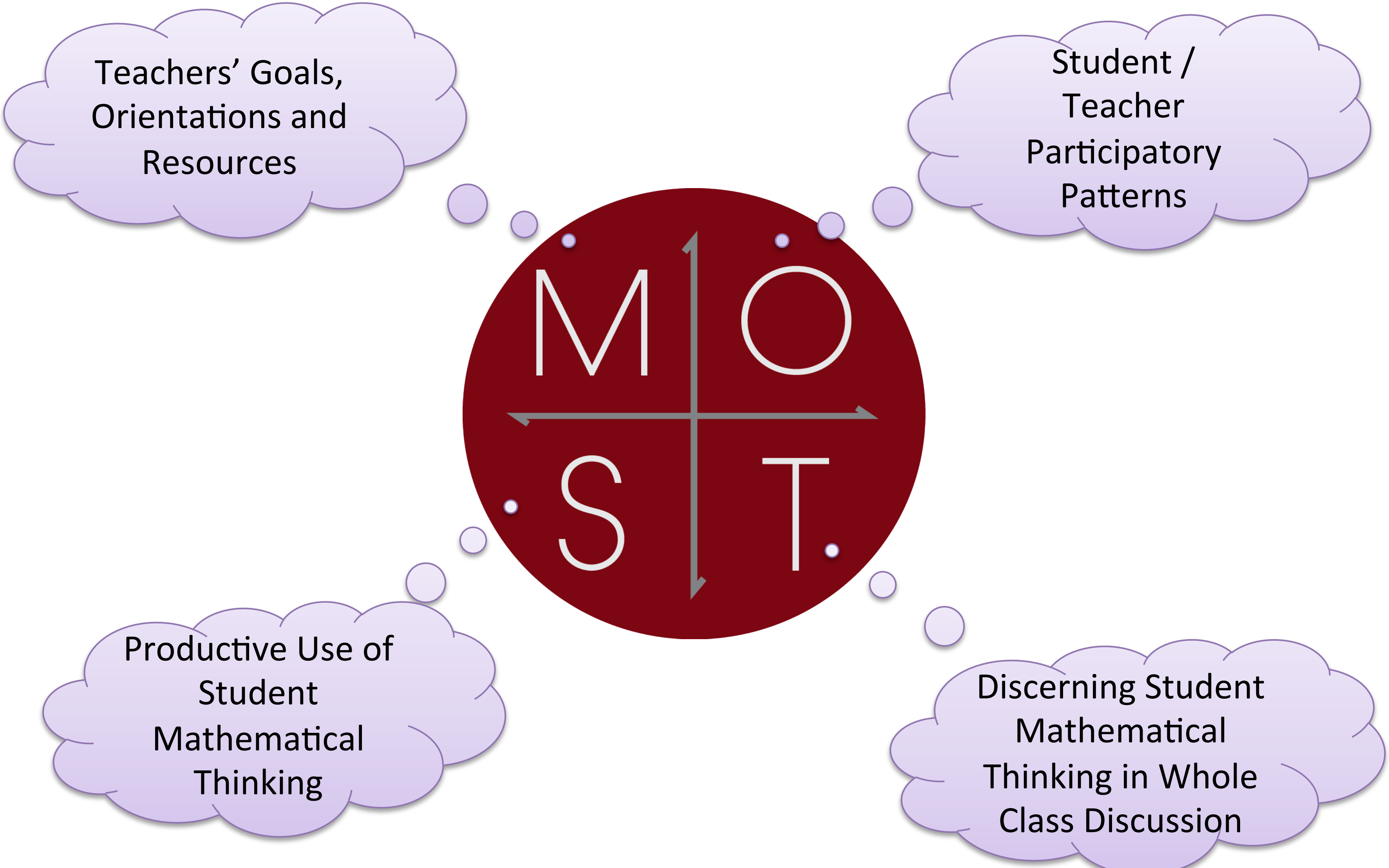
Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46*, 88-124.

MOSTs are opportunities for…

…for the teacher to make student thinking the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.

…to **build** on student thinking.

Viewing Classroom Mathematics Discourse



Methods

- 10 mathematics lessons
- 9 teachers
- California, Michigan, Mississippi, New Mexico & Utah
- Identify each instance of student mathematical thinking
- Code for MOSTs

MOST Frequencies

Teacher	Minutes in whole class Interaction	Number of Instances	Instances/ Minute	Number of MOSTs	MOSTs/ Minute	% of Instances that are MOSTs
A	25	35	1.4	8	0.3	23
B	44	227	5.2	39	0.9	17
C	27	122	4.5	22	0.8	18
D	45	206	4.6	29	0.6	14
E	48	331	6.9	78	1.6	24
F	41	176	4.3	42	1.0	24
G	35	201	5.7	33	0.9	16
H	18	113	6.3	8	0.4	7
I1	15	38	2.5	1	0.1	3
I2	11	30	2.7	5	0.5	17
Total	309	1479	4.8	265	0.9	18

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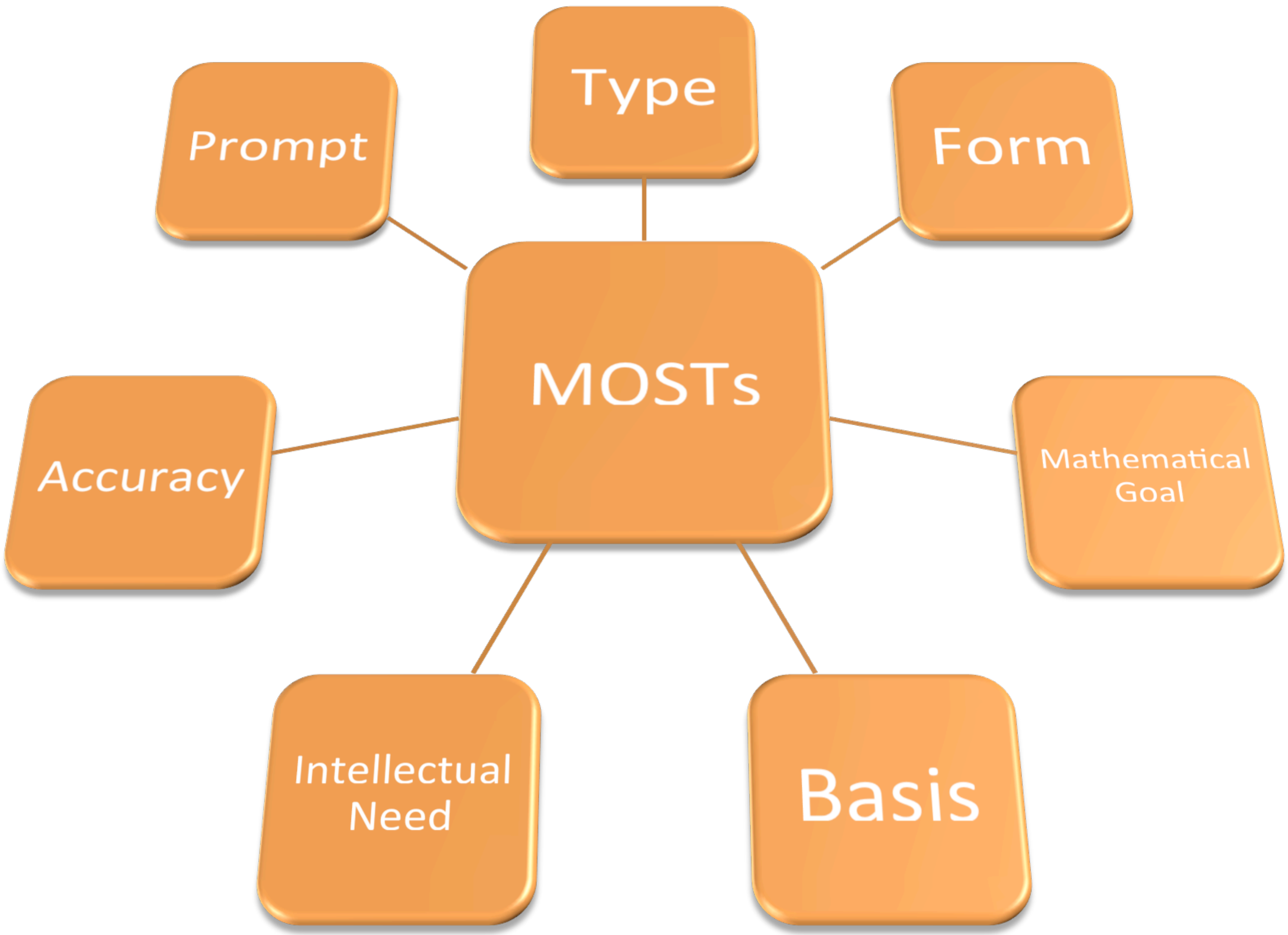
MOST Attributes



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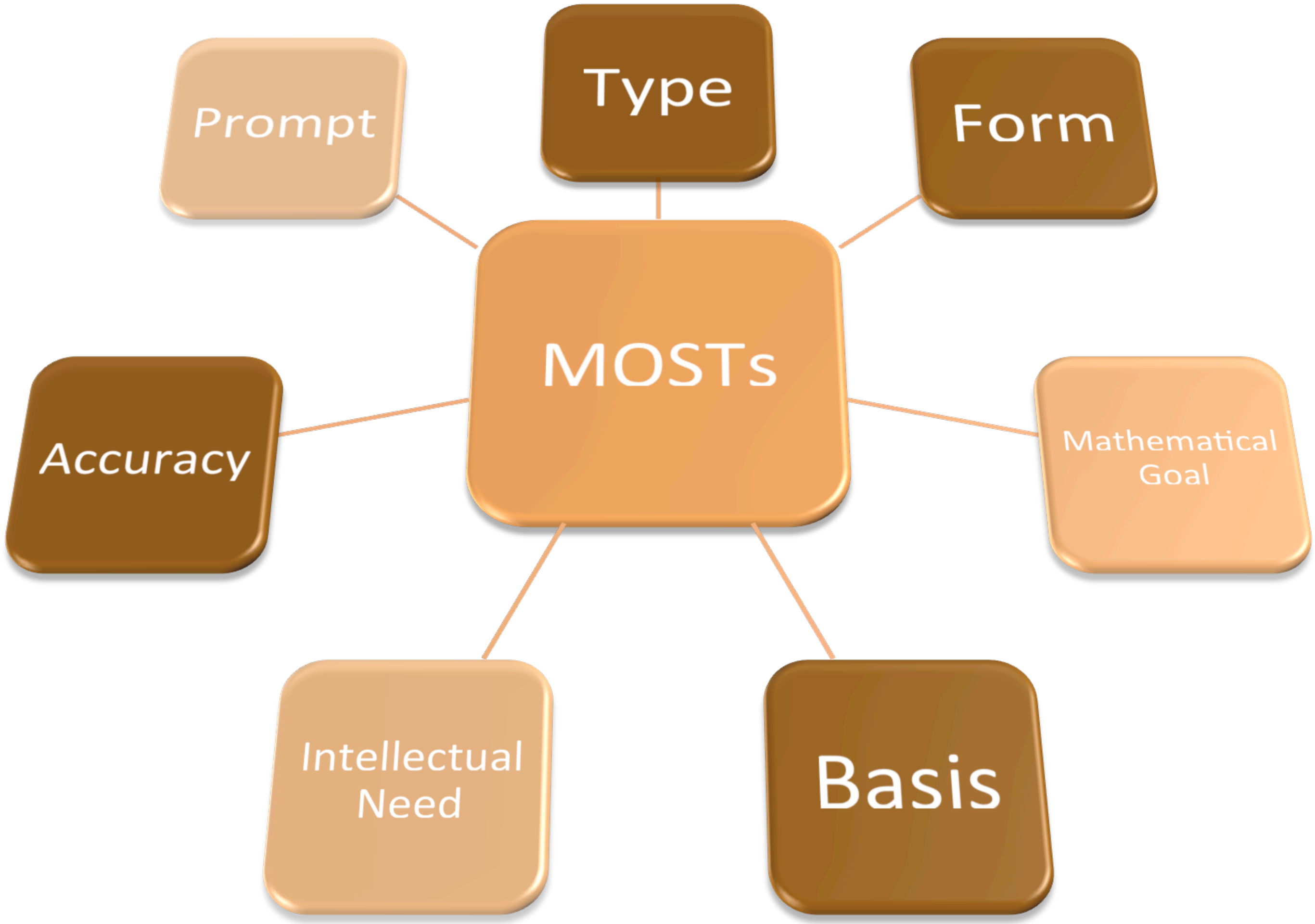
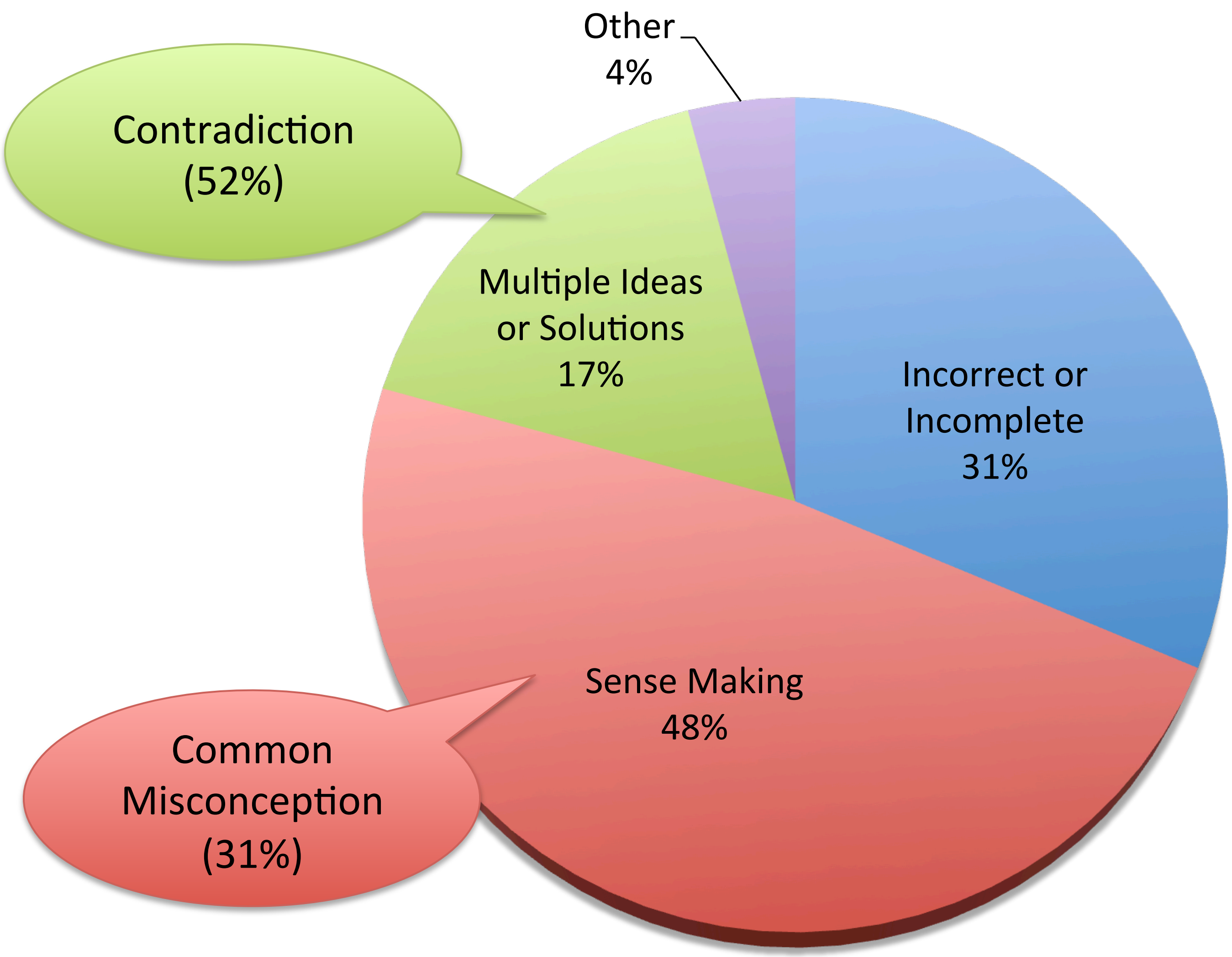
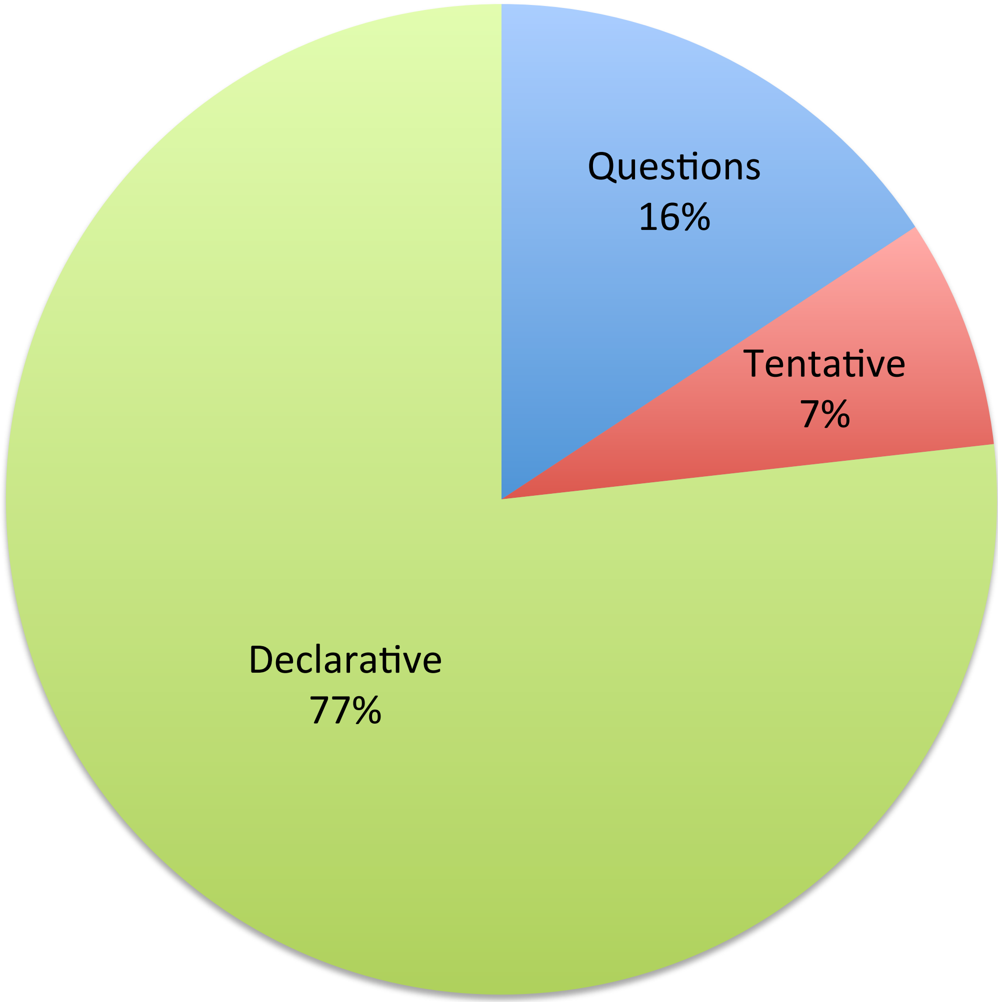


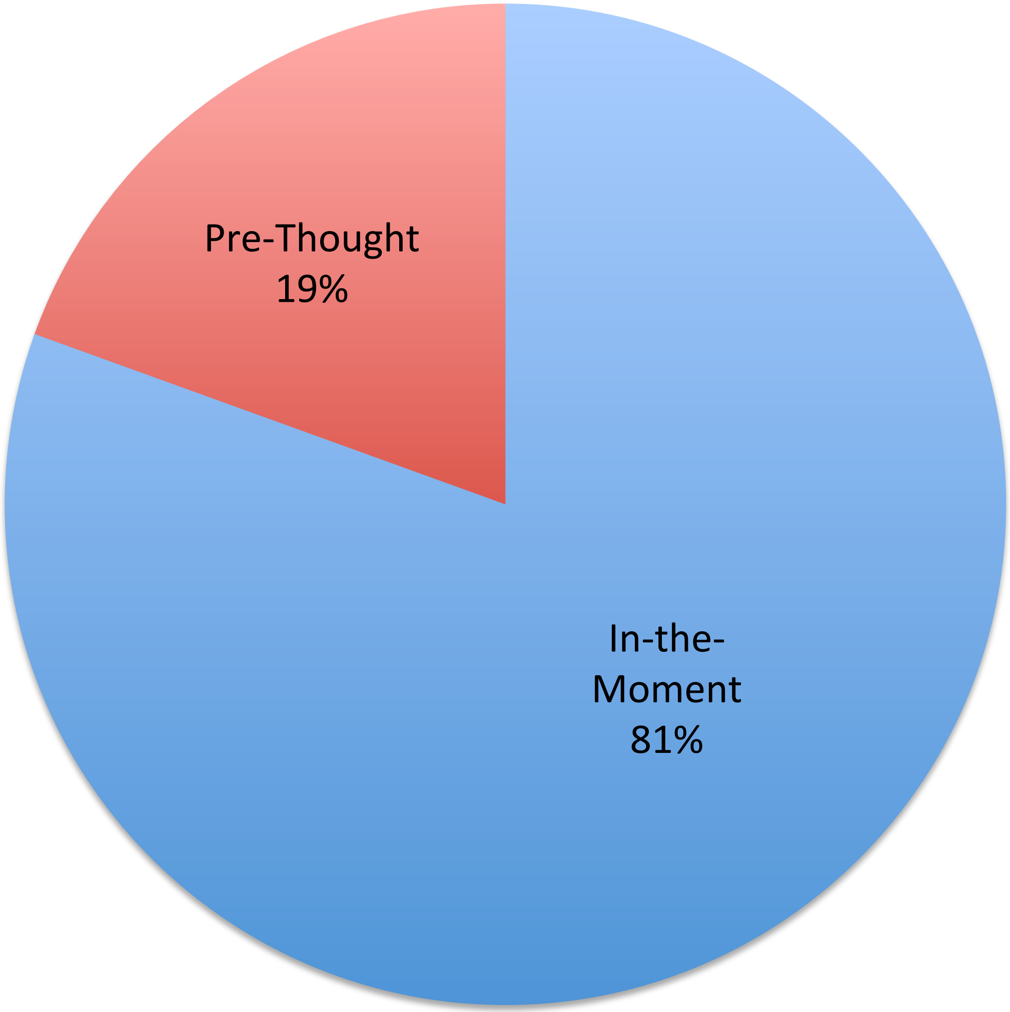
MOST Types


Other
4%
Contradiction
(52%)
Multiple Ideas
or Solutions
17%
Incorrect or
Incomplete
31%
Sense Making
48%
Common
Misconception
(31%)

MOST Forms



MOST Basis


Pre-Thought
19%
In-the-
Moment
81%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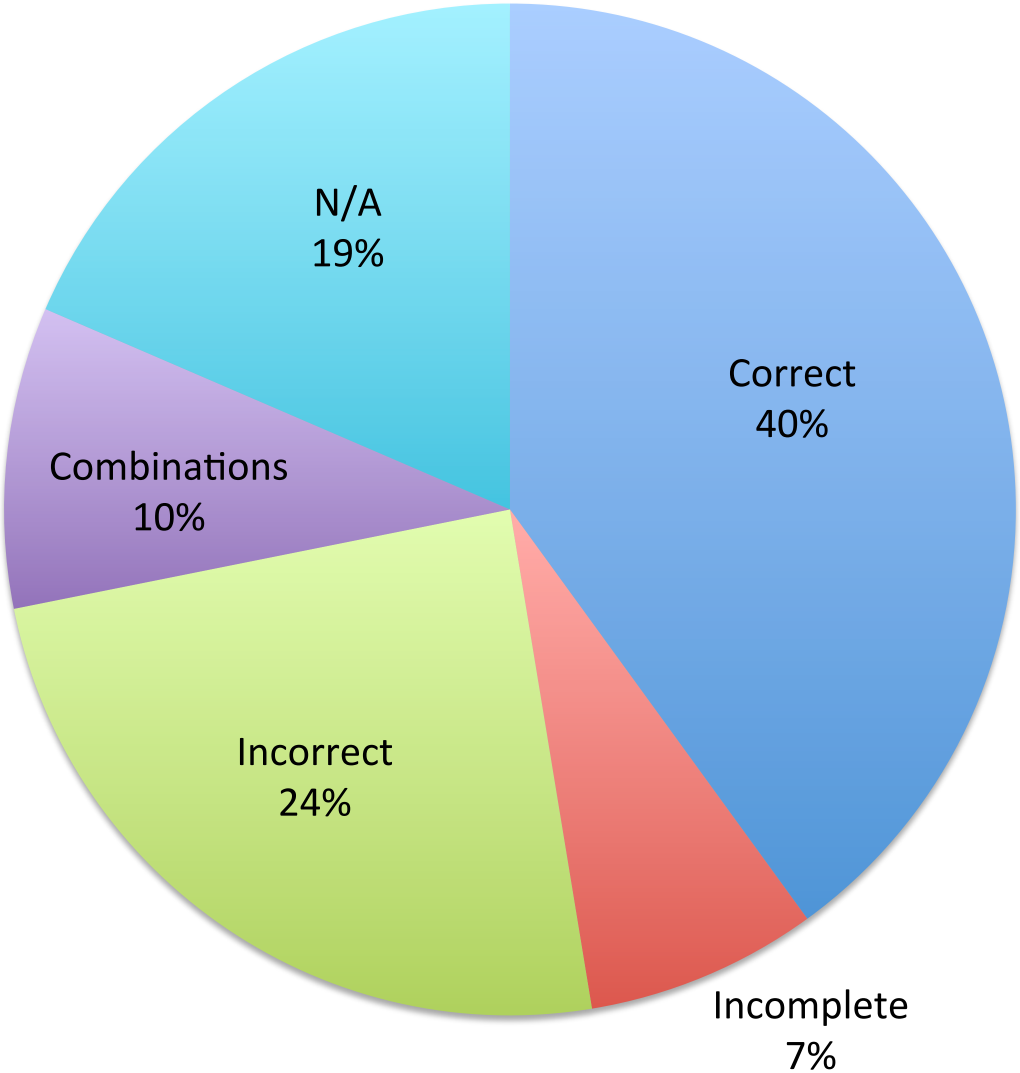
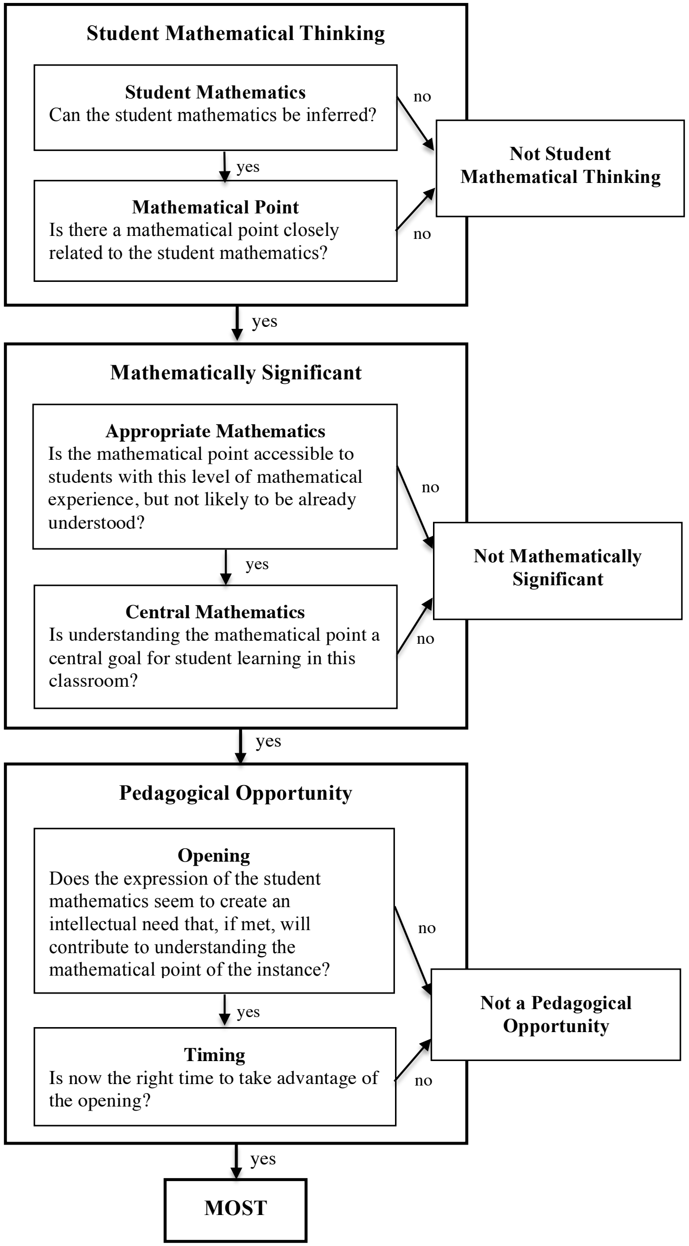
ESTABLISHED 1903

MOST Accuracy



MOST Analytic Framework



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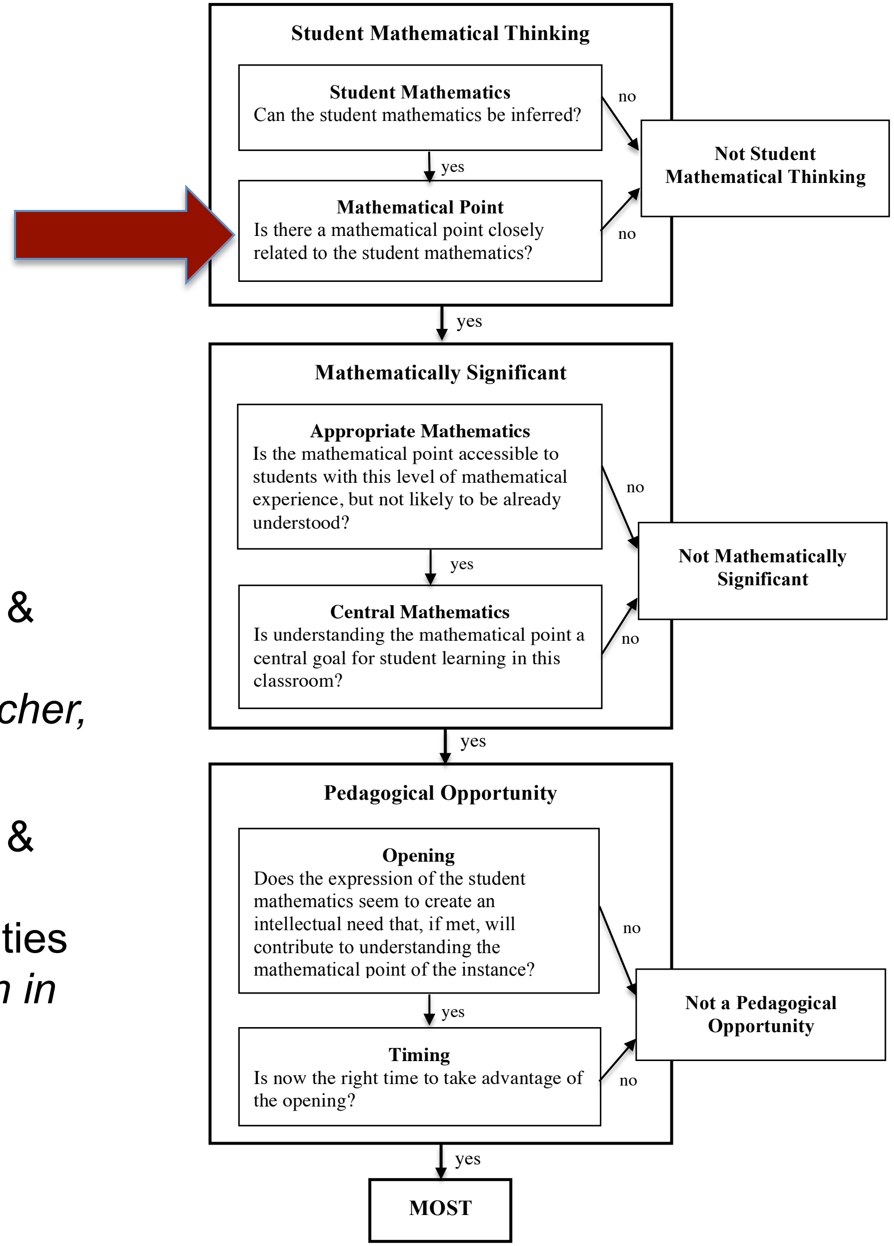

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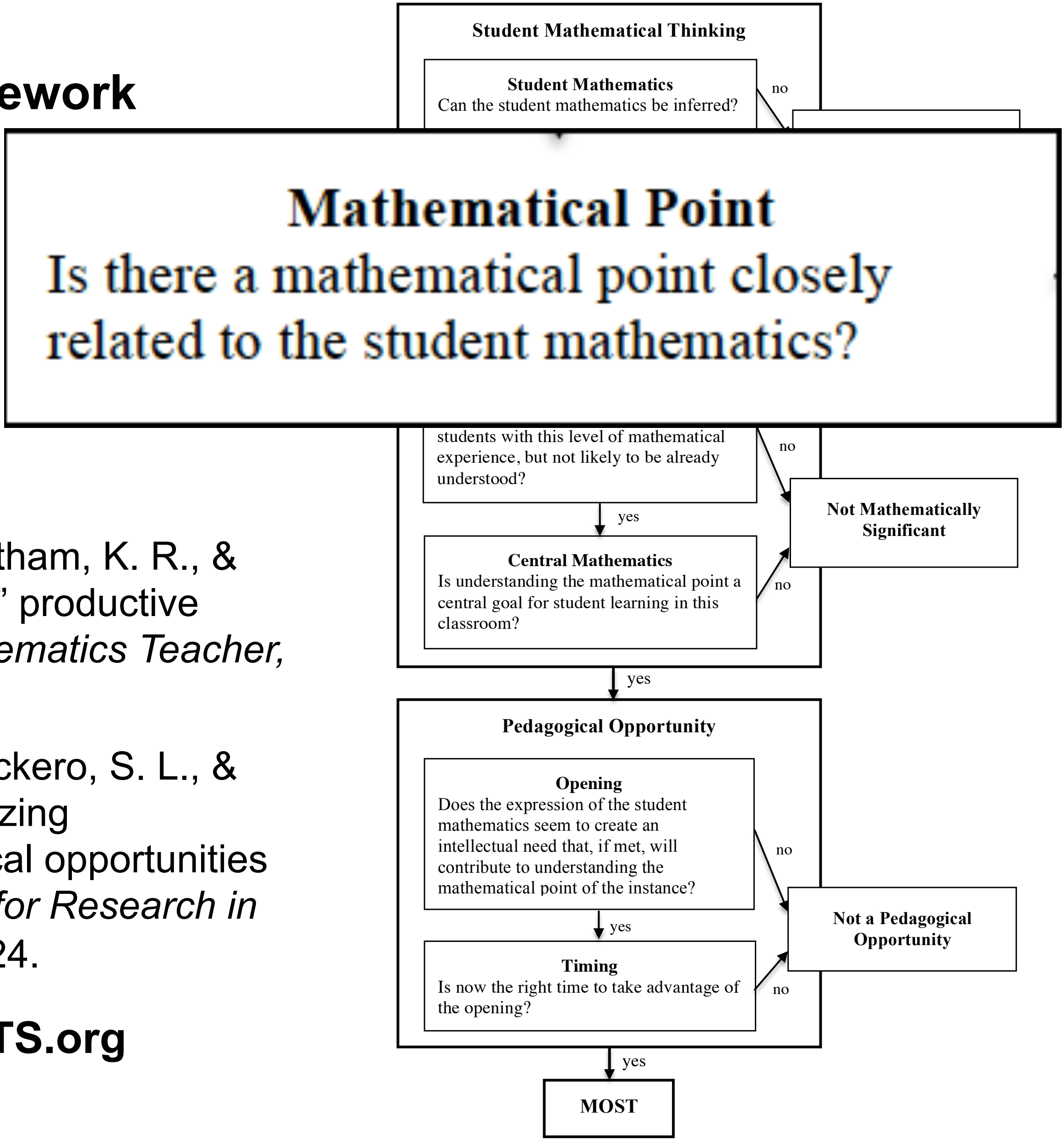

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Context

- Identifying MOSTs – *Mathematical Opportunities in Student Thinking* – high potential instances of student thinking in whole class interactions
- Thinking about how teachers can make productive use of MOSTs

Building

Teaching practice of making student thinking the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.

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Motivation

- We are interested in preparing teachers to productively use student mathematical thinking.
- An important first step is identifying what mathematical idea, if any, underlies an instance of student thinking.
- In the middle school mathematics methods course I taught last fall, we attempted to develop the preservice teachers' abilities to identify mathematical ideas in student thinking.

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An Instance of Student Thinking

A teacher in an 8th grade algebra class that had been working on writing linear equations from a table of values for a specific situation asked, “How do we find the slope given any table?” and put this (to the right) generic table of values on the board for the students to use in their explanation. A student replied, “I found the slope by subtracting the y-values in the table, 21 - 19, so the slope is 2.”

X	Y
0	15
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What mathematical idea do you think this instance of student thinking could be used to better understand?

Mathematical Point

A concise statement of a specific mathematical idea that students could learn from an instance of student thinking.

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Student Thinking: *I found the slope by subtracting the y-values in the table, 21 - 19, so the slope is 2.*

Mathematical Point: *The slope of a linear equation is* $(y_2-y_1)/(x_2-x_1)$.

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<u>Activity 1: Mathematical Point Card Sort</u>

Determine whether the statement on each card is an example or a non-example of a mathematical point

A	Addition and subtraction are inverse operations.	N	Knowing the meaning of the numerator and the denominator in a fraction.
B	What it means to multiply a fraction and a whole number.	O	Proving something is true requires proving that it is true for all possible cases.
C	When attempting to disprove a mathematical statement it is sufficient to provide a single counterexample.	P	Identify the relationship between an angle of elevation and the corresponding angle of depression.
D	Use of prime factorization to find greatest common factor of two numbers.	Q	The fraction $\frac{1}{n}$ represents one piece of a whole partitioned into n equal size pieces.
E	Mathematical models are approximations of situations.	R	Division of fractions can be visualized using both partitioning and measurement models.
F	Various forms of a line and their uses.	S	Demonstrate that two expressions are equivalent.
G	Addition and subtraction of whole numbers.	T	Two expressions are equivalent if they can be algebraically manipulated to be identical.
H	Adding fractions with a common denominator can be done by adding their numerators and keeping the common denominator because the denominator represents the size of the pieces and the numerators represent the number of pieces.	U	Notice visual and numerical patterns in the relationship between dependent and independent variables in a linear situation and describe this relationship.
I	The point-slope form of a line makes explicit a point on the line and the slope of the line.	V	Visualizing what a fraction is doing when it is being divided by another fraction.
J	Proof by counterexample.	W	Multiplication of a fraction and a whole number can be thought of as repeated addition.
K	Creating models to reflect situations.	X	When a higher vantage point A and a lower vantage point B are viewed relative to one another, the angle of elevation from B to A and the angle of depression from A to B are congruent to one another.
L	Prime factorization can be used to find the greatest common factor of two numbers.	Y	Words, equations, graphs, and tables are all valid ways to describe relationships.
M	How to get a common denominator when adding fractions.	Z	Being able to prove that a mathematical statement is true.

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G	Addition and subtraction of whole numbers.	Q	The fraction $\frac{1}{n}$ represents one piece of a whole partitioned into *n* equal size pieces.
H	Adding fractions with a common denominator can be done by adding their numerators and keeping the common denominator because the denominator represents the size of the pieces and the numerators represent the number of pieces.	U	Notice visual and numerical patterns in the relationship between the dependent and independent variables and describe this relationship.
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Concise: Are there more words included than need to be?

Specific: Is it of a small enough grain size?

	Examples		Non-Examples
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X	When a higher vantage point A and a lower vantage point B are viewed relative to one another, the angle of elevation from B to A and the angle of depression from A to B are congruent to one another.	P	Identify the relationship between an angle of elevation and the corresponding angle of depression.
Y	Words, equations, graphs, and tables are all valid ways to describe relationships.	U	Notice visual and numerical patterns in the relationship between dependent and independent variables in a linear situation and describe this relationship.

Good News

After the card sort preservice teachers were able to:

- Identify examples and non-examples
- Articulate a mathematical point for a mathematical idea

Abigail said: “Well I think [*to understand the proportional relationship from the graph*] would be a non-point.”

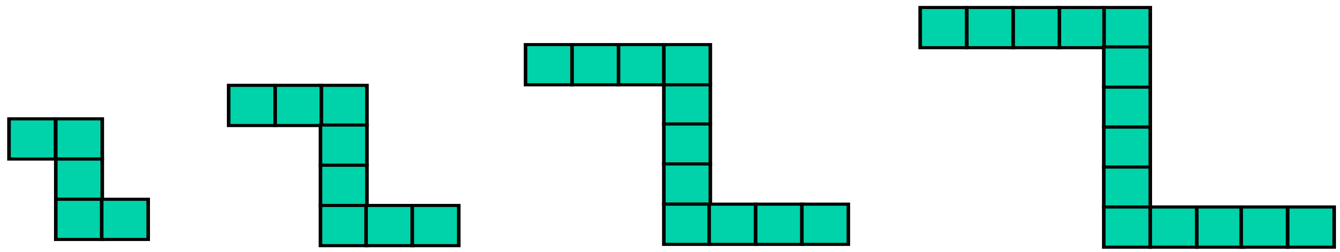
To fix the “non-point” above, Abigail suggested: “Something like *proportional relationships represented in a graph must pass through the origin.*”

Bad News

Preservice teachers had difficulty articulating mathematical points for *student thinking*.

> When Kevin was asked what mathematical point he would be working towards with a specific piece of student thinking, he said: "It was talking about the corners, where like what you could do with that, what that contributes into the problem. … I don't know like what you'd call it though. As far as that, because [14 second pause] I don't know. … Well, *deciphering the differences in patterns*. Is that a mathematical point?"

Regina's Logo Lesson Task



size 1 size 2 size 3 size 4

Assume the pattern continues to grow in the same manner.

Activity 2: Mathematical Points in Student Work

Student	Mathematical Point Possibilities
	1. A generalization requires making an argument that the pattern works for all cases.
	2. A linear relationship can be represented by multiple patterns (some more efficient than others).
	3. If a geometric pattern contains k pieces of size x, where x is the independent variable, these pieces can be represented by kx.
	4. The difference between explicit and recursive approaches.
	5. Adding an expression to itself can be represented by multiplying the expression by two.
	6. An explicit approach gives an immediate answer for any input while a recursive approach requires knowing the output for the proceeding input.
	7. In a visual representation of linear recursive situation, the figure for each step is embedded in the figure for the next step.
	8. In the linear equation $y = mx + b$, m represents the amount added each time.

Mathematical Points in Student Work (Abbreviated)

For the three pieces of Regina's Logo student work—Brian, Caitlyn, and Terrence—put the student's name next to the mathematical point that best represents the mathematics underlying the student work. Note that there will be one mathematical point possibility that does not have a corresponding student name.

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Mathematical Points in Student Work

Student	Mathematical Point Possibilities
Brian	A generalization requires making an argument that the pattern works for all cases.
Terrence	If a geometric pattern contains k pieces of size x, where x is the independent variable, these pieces can be represented by kx.
Distractor	In the linear equation $y = mx + b$, m represents the amount added each time.
Caitlyn	Adding an expression to itself can be represented by multiplying the expression by two.

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Brian	A generalization requires making an argument that the pattern works for all cases.
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Terrence	If a geometric pattern contains k pieces of size x, where x is the independent variable, these pieces can be represented by kx.
Distractor 2	The difference between explicit and recursive approaches.
Caitlyn	Adding an expression to itself can be represented by multiplying the expression by two.
Eric	An explicit approach gives an immediate answer for any input while a recursive approach requires knowing the output for the proceeding input.
Shakira	In a visual representation of linear recursive situation, the figure for each step is embedded in the figure for the next step.
Distractor 3	In the linear equation $y = mx + b$, m represents the amount added each time.

Issues we were able to address as a result of these activities

- Mathematical points must be accurate, but the student thinking they are based on may or may not be
- Mathematical points must be "close to the trunk"
- The student doesn't have to make the mathematical point
- The mathematical point is what the teacher has in their head, not something they would tell the students

Moving Towards Productive Use of Student Thinking

Ted: “[Mathematical points] allow you to use where the student’s understanding currently is in a way to move them toward the understanding of a mathematical idea you want them to be at. This allows you to build off student ideas yourself or preferably through student discussions in such a way that raises everyone’s understanding of the mathematics being discussed.”

Isabelle: “It is important to have mathematical points in mind because this focuses and guides the lesson towards highlighting important math concepts. It requires understanding of student thinking and interpreting it in order to probe student thinking in the direction toward the point.”

Observations

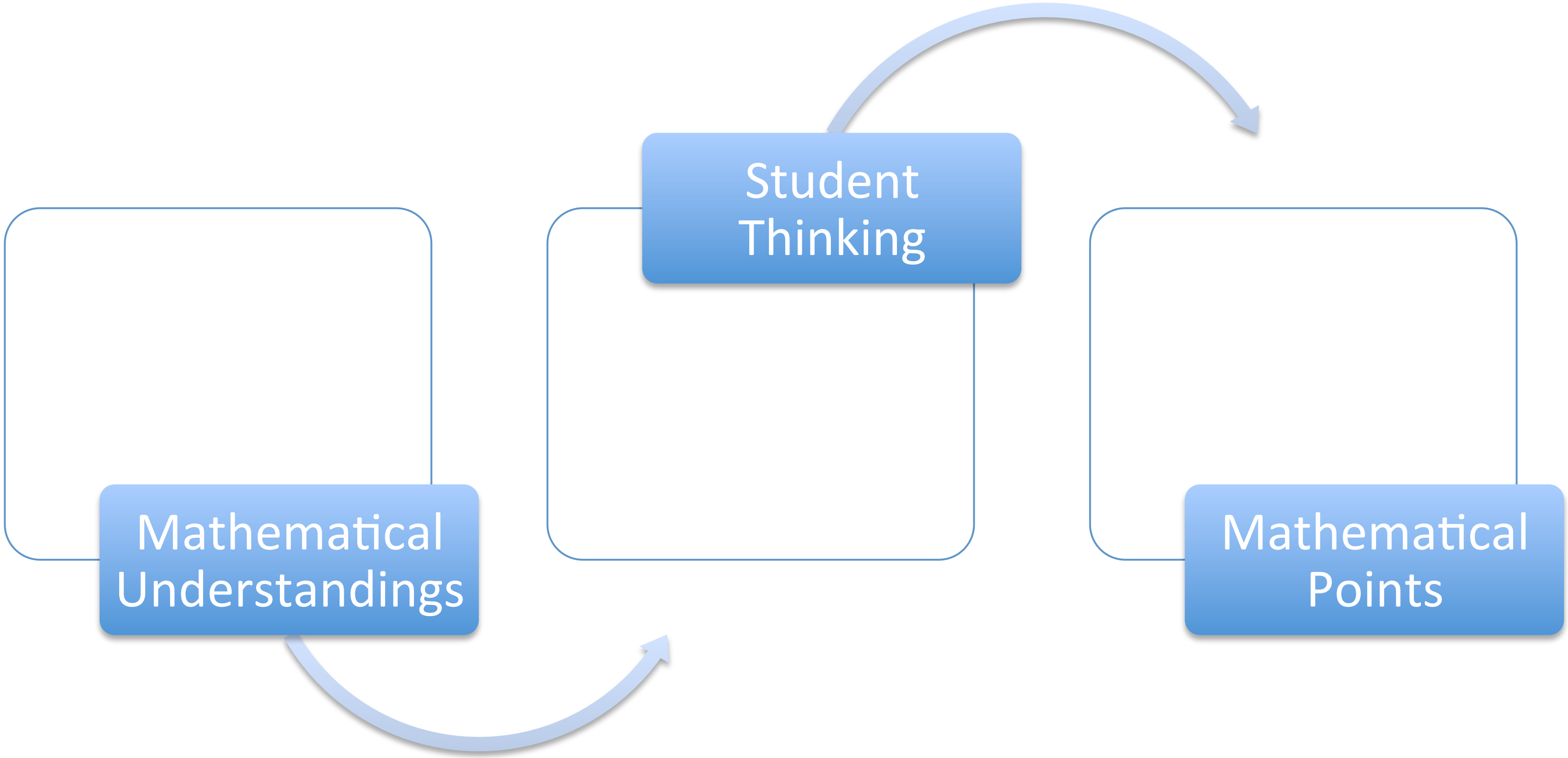
- These activities seemed to make some progress toward
  - preservice teachers seeing the importance of mathematical points in productively using student thinking
  - supporting their abilities to articulate mathematical points underlying student thinking
- Articulating mathematical points is a difficult but important teaching practice
- Next steps involve supporting preservice teachers' abilities to
  - decide which student thinking is worth pursuing
  - productively pursue that student thinking

Your thoughts

- What is the potential of identifying and conceptualizing mathematical points for an instance of student mathematical thinking as an “improving teaching” activity for inservice teachers?
- What could be done differently?

Big Ideas and Understandings
Charles, 2005

- A *Big Idea* is a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole.
- A *mathematical understanding* is an important idea students need to learn because it contributes to understanding the Big Idea.



The more a teacher knows the mathematical understandings related to the big ideas of their course, the better position they will be in to identify mathematical points in instances of student thinking that occur in their classroom.

Mathematics Teaching Practices

The Case Studies
from Diane Briar's
Opening Session

Five Practices Model

Smith & Stein, 2011

Practice 1: Anticipating

– instances of student thinking (new grain size?)

– mathematical point related to that thinking (concise?)

What does focusing on mathematical point add?

Questions or Observations?

Thanks for coming!