LeveragingMOSTs.org

Theorizing the Mathematical Point of Building on Student Mathematical Thinking

Laura R. Van Zoest
Western Michigan University

Shari L. Stockero
Michigan Technological University

Keith R. Leatham & Blake E. Peterson
Brigham Young University

Supported by
Leveraging MOSTs: Developing a Theory of Productive Use of Student Mathematical Thinking
a 4-year collaborative research project funded by the US National Science Foundation (DRL-1220141, DRL-1220357, DRL-1220148)

Using Student Mathematical Thinking in Instruction

Mathematical Opportunities in Student Thinking

- The mathematics education community has long encouraged instruction that uses students' mathematical thinking (e.g., NCTM, 1989, 2000, 2014)
- The benefits of using students' mathematical thinking in instruction have been well-documented in the areas of
  - Constructing models of student thinking and drawing on them to inform instruction (e.g., Fennema, et al., 1996; Franke & Kazemi, 2001 [CGI])
  - Using student thinking from high-cognitive demand tasks to orchestrate productive mathematics discussions (e.g., Stein & Lane, 1996; Smith & Stein, 2011 [QUASAR])
- Incorporating in-the-moment student thinking into instruction *at that moment*—what we call *the teaching practice of **building***—has untapped potential

Principles Underlying Quality Mathematics Instruction

- Mathematics is at the forefront
- Students are positioned as legitimate mathematical thinkers
- Students are engaged in sense making
- Students are working collaboratively

~extracted from NCTM, 2014, *Principles to Actions*

We see the *teaching practice of **building*** as simultaneously enacting these principles in response to student thinking.

Building on Student Mathematical Thinking

Mathematical Opportunities in Student Thinking

Building Definition**:**

Making student thinking the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.

Building Sub-Practices:

1. Make the object of consideration clear (make precise)
2. Turn the object of consideration over to the students with parameters that put them in a sense-making situation (grapple toss)
3. Orchestrate a whole-class discussion in which students collaboratively make sense of the object of consideration (orchestrate)
4. Facilitate the extraction and articulation of the **mathematical point** of the object of consideration (make explicit)

What does it take to build?

- an **instance of student thinking**: an observable student action or small collection of connected actions (such as a verbal expression combined with a gesture)
- **student mathematics (SM)**: the articulation of a reasoned inference about what the student is saying mathematically in the instance
- **mathematical point (MP)**: the articulation of the most closely related mathematical idea that can be gained from considering the instance of student thinking

Caveat

- Our goal is to improve classroom teaching in real time, but what we are proposing requires detailed attention to student thinking beyond what can be done in-the-moment of teaching.
- Preliminary evidence suggests that doing this detailed work in professional development supports teachers in doing the work in real time.

What does it take to build?

Mathematical Opportunities in Student Thinking

- an **instance of student thinking**: an observable student action or small collection of connected actions (such as a verbal expression combined with a gesture)
- **student mathematics (SM)**: the articulation of a reasoned inference about what the student is saying mathematically in the instance
- **mathematical point (MP)**: the articulation of the most closely related mathematical idea that can be gained from considering the instance of student thinking

Example

Context/Instance

In an algebra lesson on solving simple linear equations, the class is discussing how to solve for m in the equation $m - 12 = 5$ and a student responds, "**Subtract 12 from both sides**."

Student Mathematics (SM)

To solve for m in the equation $m - 12 = 5$, subtract 12 from both sides of the equation.

Which is the mathematical point (MP) for this student mathematics (SM)?

Solving linear equations.	An integer and its opposite are the same distance from zero on the number line.	Adding a number and subtracting that same number are inverse operations.	Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation.

Mathematical Point (MP): the articulation of the most closely related mathematical idea that can be gained from considering the instance of student thinking

Example

Mathematical Opportunities in Student Thinking

Context/Instance

In an algebra lesson on solving simple linear equations, the class is discussing how to solve for m in the equation $m - 12 = 5$ and a student responds, "**Subtract 12 from both sides.**"

Student Mathematics (SM)

To solve for m in the equation $m - 12 = 5$, subtract 12 from both sides of the equation.

Which is the mathematical point (MP) for this student mathematics (SM)?

Not well-specified

Solving linear equations.

An integer and its opposite are the same distance from zero on the number line.

Adding a number and subtracting that same number are inverse operations.

Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation.

Mathematical Point (MP): the articulation of the most closely related mathematical idea that can be gained from considering the instance of student thinking

Example

Context/Instance

In an algebra lesson on solving simple linear equations, the class is discussing how to solve for m in the equation $m - 12 = 5$ and a student responds, "**Subtract 12 from both sides.**"

Student Mathematics (SM)

To solve for m in the equation $m - 12 = 5$, subtract 12 from both sides of the equation.

Which is the mathematical point (MP) for this student mathematics (SM)?

Not well-specified

Solving linear equations.

Cannot get there

...ger and ...e opposite are the same distance from zero on the number line.

Adding a number and subtracting that same number are inverse operations.

Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation.

Mathematical Point (MP): the articulation of the most closely related mathematical idea that can be gained from considering the instance of student thinking

Example

Mathematical Opportunities in Student Thinking

Context/Instance

In an algebra lesson on solving simple linear equations, the class is discussing how to solve for m in the equation $m - 12 = 5$ and a student responds, "**Subtract 12 from both sides.**"

Student Mathematics (SM)

To solve for m in the equation $m - 12 = 5$, subtract 12 from both sides of the equation.

Which is the mathematical point (MP) for this student mathematics (SM)?

Well-specified truth that could be gained

Not well-specified

Solving linear equations.

Cannot get there

...ger and ...e opposite are the same distance from zero on the number line.

Adding a number and subtracting that same number are inverse operations.

Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation.

Mathematical Point (MP): the articulation of the most closely related mathematical idea that can be gained from considering the instance of student thinking

Example

Mathematical Opportunities in Student Thinking

Context/Instance

In an algebra lesson on solving simple linear equations, the class is discussing how to solve for m in the equation $m - 12 = 5$ and a student responds, "**Subtract 12 from both sides.**"

Student Mathematics (SM)

To solve for m in the equation $m - 12 = 5$, subtract 12 from both sides of the equation.

Which is the mathematical point (MP) for this student mathematics (SM)?

Well-specified truth that could be gained

Not well-specified

Solving linear equations.

Cannot get there

ger and opposite are the same distance from zero on the number line.

Not the closest

a umber and subtracting that same number are inverse operations.

Math Point (MP)

Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation.

Mathematical Point (MP): the articulation of the most closely related mathematical idea that can be gained from considering the instance of student thinking

Mathematical Point (MP)

The *mathematical understanding* (MU)* that

1. students could gain from considering a particular instance of student thinking
2. is most closely related to the student mathematics of the thinking

*well-specified statement of a mathematical truth

Another Example

Mathematical Opportunities in Student Thinking

Context/Instance

In an introductory lesson on adding fractions with like denominators, a student **writes "2/5 + 1/5 = 3/10" on the board**.

Student Mathematics (SM)

2/5 + 1/5 = 3/10

Which is the mathematical point (MP) for this student mathematics (SM)?

Every fraction/ ratio can be represented by an infinite set of different but equivalent fractions/ratios.	Adding fractional pieces of the same size changes the number of pieces, but not the size of the pieces.	Adding two quantities means combining the amounts together.	How to get a common denominator when adding fractions.

Mathematical Point ~ The mathematical understanding (well-specified statement of a mathematical truth) that (1) students could gain from considering a particular instance of student thinking; and (2) is most closely related to the student mathematics of the thinking.

Another Example

Mathematical Opportunities in Student Thinking

Context/Instance

In an introductory lesson on adding fractions with like denominators, a student **writes "2/5 + 1/5 = 3/10" on the board**.

Student Mathematics (SM)

2/5 + 1/5 = 3/10

Which is the mathematical point (MP) for this student mathematics (SM)?

Every fraction/ ratio can be represented by an infinite set of different but equivalent fractions/ratios.	Adding fractional pieces of the same size changes the number of pieces, but not the size of the pieces.	Adding two quantities means combining the amounts together.	Not well-specified How to get a common denominator when adding fractions.

Mathematical Point ~ The mathematical understanding (well-specified statement of a mathematical truth) that (1) students could gain from considering a particular instance of student thinking; and (2) is most closely related to the student mathematics of the thinking.

Mathematical Opportunities in Student Thinking

Another Example

Context/Instance

In an introductory lesson on adding fractions with like denominators, a student **writes “2/5 + 1/5 = 3/10” on the board**.

Student Mathematics (SM)

2/5 + 1/5 = 3/10

Which is the mathematical point (MP) for this student mathematics (SM)?

Mathematical Understanding (MU)

Every fraction/ratio can be represented by an infinite set of different but equivalent fractions/ratios.	Adding fractional pieces of the same size changes the number of pieces, but not the size of the pieces.	Adding two quantities means combining the amounts together.	Not well-specified ow to get a common denominator when adding fractions.

Mathematical Point ~ The mathematical understanding (well-specified statement of a mathematical truth) that (1) students could gain from considering a particular instance of student thinking; and (2) is most closely related to the student mathematics of the thinking.

Another Example

M O S T
Mathematical Opportunities in Student Thinking

Context/Instance

In an introductory lesson on adding fractions with like denominators, a student **writes “2/5 + 1/5 = 3/10” on the board**.

Student Mathematics (SM)

2/5 + 1/5 = 3/10

Which is the mathematical point (MP) for this student mathematics (SM)?

MUs that can be gained

Cannot get there Ev… …ion/ …can be represented by an infinite set of different but equivalent fractions/ratios.	Adding fractional pieces of the same size changes the number of pieces, but not the size of the pieces.	Adding two quantities means combining the amounts together.	Not well-specified …w to get a common denominator when adding fractions.

Mathematical Point ~ The mathematical understanding (well-specified statement of a mathematical truth) that (1) students could gain from considering a particular instance of student thinking; and (2) is most closely related to the student mathematics of the thinking.

Another Example

Mathematical Opportunities in Student Thinking

Context/Instance

In an introductory lesson on adding fractions with like denominators, a student **writes “2/5 + 1/5 = 3/10” on the board**.

Student Mathematics (SM)

2/5 + 1/5 = 3/10

Which is the mathematical point (MP) for this student mathematics (SM)?

MUs that can be gained

Cannot get there

Ev… …ion/ …can be represented by an infinite set of different but equivalent fractions/ratios.

Math Point (MP)

Adding fractional pieces of the same size changes the number of pieces, but not the size of the pieces.

Not the closest

…two …uantities means combining the amounts together.

Not well-specified

…ow to get a common denominator when adding fractions.

Mathematical Point ~ The mathematical understanding (well-specified statement of a mathematical truth) that (1) students could gain from considering a particular instance of student thinking; and (2) is most closely related to the student mathematics of the thinking.

Some things to keep in mind

Mathematical Opportunities in Student Thinking

- MPs are always mathematically correct statements, although the corresponding student mathematics may or may not be.
- An MP exists only in relation to a specific instance of student thinking.
- To be an MP, the mathematical understanding must be gained from considering the student thinking itself.
- Not all instances of student thinking give rise to an MP.
- There are acceptable variations in the articulation of SMs, mathematical understandings, and MPs.

Why Mathematical Point?

Mathematical Opportunities in Student Thinking

- Instructional practices that use student thinking lead to improved learning (NCTM, 2014)
- Identifying the MP in student thinking supports effective decision making about which student thinking to use (MOST Analytic Framework)
- Having the MP in mind supports facilitating productive discussion of the student thinking by the class (Building)

Mathematical Point Examples and Non-Examples

Mathematical Point ~ The mathematical understanding (well-specified statement of a mathematical truth) that (1) students could gain from considering a particular instance of student thinking; and (2) is most closely related to the student mathematics of the thinking.

Context	Student Mathematics (SM)	Well-Specified Statement of a Mathematical Truth			Not Well-Specified
		Could be gained from considering SM		Could not be gained from considering SM	
		Mathematical Point	Not Closest to SM		
In an algebra lesson on solving linear equations, the class is discussing how to solve for m in the equation $m - 12 = 5$ and a student responds, "Subtract 12 from both sides."	To solve for m in the equation $m - 12 = 5$, subtract 12 from both sides of the equation.	Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation.	Adding a number and subtracting that same number are inverse operations.	An integer and its opposite are the same distance from zero on the number line. (Charles, 2005, p. 18)	Solving linear equations.
In a beginning algebra lesson on representing linear situations with equations, the equation (t*2.5)+25 = m is on the board. A student says, "you don't need the parentheses."	In the equation (t*2.5)+25 = m, the parentheses around t*2.5 are optional.	Parentheses are necessary when the intended order of operations differs from the conventional order of operations.	The commutative property applies to addition and multiplication but not subtraction and division. (Charles, 2005, p. 16)	The nature of the quantities in a relationship determines what values of the input and output quantities are reasonable. (Charles, 2005, p. 18)	Order of operations.
In an introductory lesson on adding fractions with like denominators, a student writes 2/5 + 1/5 = 3/10 on the board.	2/5 + 1/5 = 3/10.	Adding fractional pieces of the same size changes the number of pieces, but not the size of the pieces.	Adding two quantities means combining the amounts together.	Every fraction/ratio can be represented by an infinite set of different but equivalent fractions/ratios. (Charles, 2005, p. 18)	How to get a common denominator when adding fractions.
On day two of a unit on solving simple linear equations, the teacher writes x = 3 as the solution to $x + 2 = 5$, and a student remarks, "Hey, wait a minute, yesterday you said x equals two!"	Yesterday x equaled 2 and today x equals 3.	A letter can be used to represent an unknown quantity in an equation and can represent different quantities for different equations.	Letters can be used to represent unknown quantities, varying quantities, and arguments for a function.	Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation.	The meaning of variable.

Discussion

Mathematical Opportunities in Student Thinking

LeveragingMOSTs.org