# TEACHERS' PERCEPTIONS OF PRODUCTIVE USE OF STUDENT MATHEMATICAL THINKING

Keith R. Leatham[1], Laura R. Van Zoest[2], Shari L. Stockero[3], Blake E. Peterson[4]

[1]Brigham Young University, [2]Western Michigan University
[3]Michigan Technological University, [4]Brigham Young University

*We argue that the teaching practice of productively using student mathematical thinking [PUMT] needs to be better conceptualized for the construct to gain greater traction in the classroom and in research. We report the results of a study wherein we explored teachers' perceptions of PUMT. We interviewed mathematics teachers and analysed these interviews using and refining initial conjectures about the process teachers might go through in learning PUMT. We found that teachers' perceptions of PUMT ranged from valuing student participation, to valuing student mathematical thinking, to using that thinking in a variety of ways related to eliciting, interpreting and building on that thinking.*

## INTRODUCTION

Instruction that meaningfully incorporates students' mathematical thinking is widely valued within the mathematics education community (e.g., NCTM, 2000, 2007). Past research has suggested both the benefits of instruction that incorporates student mathematical thinking to develop mathematical ideas (e.g., Fennema, et al., 1996; Stein & Lane, 1996), and the challenges of learning about and enacting such instruction (e.g., Ball & Cohen, 1999; Sherin, 2002). One reason for these challenges may be the under conceptualization of *the teaching practice of productively using student mathematical thinking* [PUMT].

The literature uses multiple terms, and the same terms in multiple ways, to describe PUMT. For example, some (e.g., Franke & Kazemi, 2001; Peterson & Leatham, 2009) talk of teachers *using* student mathematical thinking. Others (e.g., Hill, Ball, & Schilling, 2008; Leatham, Peterson, Stockero, & Van Zoest, 2014) discuss teachers *building* on student mathematical thinking, and still others (e.g., Feiman-Nemser & Remillard, 1996; Lampert, et al., 2013) refer to students *attending to* the mathematical thinking of others. Thus, although many advocate teachers *being* "*responsive to* students and… their understanding" (Remillard, 1999, p. 331), the nature of such responses is ill defined.

This imprecision in language causes challenges when supporting teachers in developing PUMT, leaving them with multiple, and sometimes unhelpful, interpretations of the practice. This imprecision also hinders productive discourse within the research community and inhibits researchers from building on each other's work. Our broader work on PUMT is designed to support teachers in developing this critical practice; thus we chose as participants practicing teachers so that we could use

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 73-80. Vancouver, Canada: PME.

their thinking to begin to address these imprecision-related challenges. Our goal is to better understand the multiple interpretations of PUMT that teachers have developed, and to initiate a discussion about what the mathematics teacher education field means by PUMT. Specifically, we investigated the question, "What are teachers' perceptions of productive use of student mathematical thinking during whole class discussion?"

## THEORETICAL PERSPECTIVES

For us productive use of student mathematical thinking requires first that one honor students as legitimate creators of mathematics. In addition, productive use in a mathematics classroom must be in the service of facilitating the learning of significant mathematics. Finally, we use "use" in the immediate sense of a teacher orchestrating student learning during a lesson. Productive use of student mathematical thinking "engages students in making sense of mathematical ideas that have originated with students—that is, it builds on student mathematical thinking by making it the object of rich mathematical discussion" (Leatham et al., 2014, p. 5). For example, suppose students in a pre-algebra class are discussing how to solve the equation $m - 12 = 5$ and someone in the class suggests subtracting 12 from both sides. A teacher could productively use this student mathematical thinking by pursuing it with the class and making sense of the outcome, all in the service of facilitating better understanding of the use of inverse operations to isolate variables when solving linear equations. (See Leatham et al., 2014 for further elucidation of this and other such examples.)

As we have already argued, enacting practices related to productively using student mathematical thinking is complex. As we have studied novice and expert teachers' attempts to enact this practice (e.g., Peterson & Leatham, 2009; Van Zoest, Stockero & Kratky, 2010) we have developed conjectures about a *hypothetical learning process* [HLP] (Simon, 1995) related to PUMT. That is, it seems as though there are critical stages that build somewhat linearly on one another as a teacher develops PUMT (see Table 1). In professional development work, the HLP would combine with the goal of developing PUMT and with learning activities to form a *hypothetical learning trajectory* [HLT] (Simon, 1995).

Although this study contributes to research on teachers' beliefs, we use the somewhat weaker term "perceptions" here because of the nature of the data collection and analysis. We use the term "perception" to mean, in essence, "initial reaction," and recognize that perceptions are part of complex sensible belief systems (Leatham, 2006). Thus we expect that teachers may have more to say about these issues if they were explored in greater depth, and we make no claim to have sufficient data to infer deeper held beliefs. Initial reactions are very interesting, however, when looked at across a group of individuals because these commonalities can be construed, to some degree, as a "common wisdom" or "common viewpoint" (Leatham, 2009). Thus studying teachers' perceptions will provide initial insights into the ways they conceptualize productive use of student mathematical thinking.

| Hypothetical Learning Process for PUMT |
|---|
| **Reject Active Student Participation** – Teachers do not see the value of students being actively engaged during instruction. |
| **Value Student Participation –** Teachers want students to be actively engaged during instruction. |
| **Value Student Mathematical Thinking** – Teachers view students as capable of diverse legitimate ways of viewing and doing mathematics. |
| **Elicit Student Mathematical Thinking** – Teachers actively provide opportunities for students to share their mathematical thinking publicly. |
| **Interpret Student Mathematical Thinking –** Teachers conscientiously attend to and make sense of the mathematical thinking that is being shared. |
| **Build on Student Mathematical Thinking** – Teachers make student mathematical thinking the object of consideration in order to engage students in making sense of that thinking to better understand an important mathematical idea. (Teachers refine this practice first with individuals, then with small groups, and eventually in whole-class settings.) |

Table 1: Hypothetical learning process for developing *the teaching practice of productively using student mathematical thinking* [PUMT].

## METHODS

Our participants were 14 mathematics teachers (6 female and 8 male) with 1 to over 20 years of experience teaching a variety of mathematics courses in grades 6-12. In order to explore teachers' perceptions of productive use of student mathematical thinking we developed an interview protocol wherein we asked each teacher to sort a collection of cards describing teacher moves one might associate with classroom discourse (e.g., "get students' ideas out there for the class to consider and discuss," "juxtapose two student ideas that differ in an important mathematical way," "repeat an important student comment"). We compiled these teacher moves from the literature, from our own experience, and from an informal survey of mathematics education colleagues that asked them to describe what it meant to build on student thinking. We asked the participants to sort the moves along a continuum, from least to most productive use of student thinking during whole-class discussion, thinking aloud as they did so. We further prompted them to explain their reasoning or describe the criteria they seemed to be applying in making their decisions as they sorted the cards. We ended the interview by asking the participants what characteristics they saw as encapsulating the moves they placed at the top (as well as the bottom) of the continuum. Prior to conducting the 14 interviews we conducted two pilot interviews and made minor revisions to the protocol. All interviews were videotaped, with the video focused on the interviewees' sorting of the cards.

Initial analysis consisted of watching and writing brief summaries for each interview, in which we attempted to capture the essence of each teacher's overall perception of productive use of student thinking. Based on these summaries and on our initial learning trajectory (see Table 1) we developed a coding framework of potential perceptions and types of uses of student thinking and returned to the data to systematically code the interviews for evidence of these perceptions and uses (or for the emergence of others). We applied this framework to the interviews (refining and reapplying as appropriate) from six teachers who were selected to be representative of the range of perceptions based on analysis of the initial summaries. We then asked the following questions of the data: What are teachers' perceptions of productive use of student thinking? To what extent do those perceptions align with the PUMT HLP? Our answers to these questions make up the results section of the paper.

## RESULTS

Initial analysis of the interviews revealed a variety of ways that teachers thought about PUMT, including different uses of student thinking during instruction. Further analysis revealed that types of use seemed to align in interesting ways with our conjectures about stages of the PUMT HLP (see Table 2). We thus organize this results section around these stages. As we discuss the stages we provide examples from the data to illustrate the participants' associated perceptions.

| PUMT HLP | Type of Use |
|---|---|
| Reject Active Student Participation | |
| Value Student Participation | |
| Value Student Mathematical Thinking | |
| Elicit Student Mathematical Thinking | Engagement<br>Replacement<br>Validation |
| Interpret Student Mathematical Thinking | Assess<br>Clarify<br>Launch |
| Build on Student Mathematical Thinking | Pondering<br>Establishing<br>Extracting |

Table 2: Conjectured relationship between the PUMT HLP and various types of use.

Before beginning our discussion of the stages on the HLP, it is important to note that an individual teacher may be functioning in several stages simultaneously. This multiplicity can be a reflection of a transition or a result of contextual factors. For example, some teachers' perceptions about productive use of student mathematical thinking were tied to the level of student (advanced vs. remedial, middle school vs. high school) or to school factors (pressure to prepare for high-stakes tests vs. freedom to vary the curriculum). These nuances are not our focus here, but deserve attention in future research.

**Non-Use Stages**

The first three stages of the PUMT HLP do not involve incorporating students' mathematical thinking into instruction. At the first stage, *Reject Active Student Participation*, teachers do not see the value of students being actively engaged during instruction. Instead, they consider the students as receivers of knowledge that the teacher presents to them. Teachers at the second stage, *Value Student Participation*, place a high regard on student participation, but in a way that seems to have little to do with the mathematical content of that participation. For example, one teacher wanted his students to understand that, "realistically, you might not use… any of these formulas in what you are going to do in life, but if you can learn to be a thinker… then that's going to be of great benefit." For this teacher participation through thinking yielded an important outcome regardless of the content of that thinking. At the third stage, *Value Student Mathematical Thinking*, teachers view students as capable of diverse legitimate ways of viewing and doing mathematics, but do not purposefully incorporate that thinking into instruction.

**Elicit Student Mathematical Thinking**

Teachers at the *Elicit* stage actively provide opportunities for students to share their mathematical thinking publicly. We have identified three types of use at this stage (not related hierarchically): (a) *Engagement*—The teacher elicits student mathematical thinking so that students will feel that they are an important part of the lesson and so that, by seeing others so engage, they will want to similarly participate. For example, one teacher indicated that any move that could elicit student mathematical thinking provided evidence that students were engaged and "trying to get the student involved is the most important thing. Everything else is secondary." (b) *Validation*—The teacher elicits student mathematical thinking to create an opportunity to provide positive feedback for students so they feel good about themselves. One teacher explained that "acknowledging that you are thinking is important because that gives you positive reinforcement." (c) *Replacement*—The teacher elicits student mathematical thinking in such a way that students say what the teacher wanted said. For example, teachers might share a student solution to a problem rather than working an example themself. Or, instead of making a statement teachers might ask a question (simple or fill-in-the-blank) so that student responses say what they would have said.

**Interpret Student Mathematical Thinking**

Teachers at the *Interpret* stage conscientiously attend to and make sense of the thinking that is being shared during their instruction. Three types of use (again not related hierarchically) were identified at this stage: (a) *Assess*—The teacher makes sense of the student mathematical thinking to determine whether given ideas are sufficiently understood to inform subsequent instruction. They may share this assessment with students, thus informing students about the correctness of their thinking. One teacher explained, "if they can verbalize how they are thinking about it then I actually get a better idea that they actually do know what is going on." (b) *Clarify*—The teacher makes sense of the student mathematical thinking and shares their own interpretation with the class with the intent to clarify the content of that thinking for the class. Some ways a teacher might clarify include adding mathematical language to a student comment, making a connection between the student thinking and a mathematical idea, and highlighting the importance of the thinking. (c) *Launch*—The teacher makes sufficient sense of the student mathematical thinking to see a connection to something they want to come out in the lesson. They then make the connection as a segue to making their point. As one teacher indicated, it is valuable to "give them suggestions about how they could advance their thinking about the mathematics, rather than just acknowledge *that* they are thinking."

### Build on Student Mathematical Thinking

Teachers at the *Build* stage make student thinking the object of consideration in order to engage students in making sense of that thinking to better understand an important mathematical idea. There are three types of use connected to this stage: (a) *Pondering*—The teacher invites the class to think about the student mathematical thinking. For example, the teacher could give students a few moments to digest an idea before moving on. One teacher indicated that a major goal in having students share their ideas is to "have the class think about them." (b) *Establishing*—The teacher creates the space for the class to make sense of the student mathematical thinking and come to a mutual understanding of what was said or meant. For example, one teacher described how they "could have the student actually write what they just said and see if… the rest of the class could apply what the other student just said to the current problem they are working on." Another teacher spoke of the value of having students "convince the other person what you're thinking or try to understand the other idea." (c) *Extracting*—The teacher orchestrates a discussion that leads to a mutual understanding of the student mathematical thinking and helps the class to see the underlying mathematics that the student thinking embodies. For example, one teacher felt that it was extremely productive to elicit a variety of student ideas and "ask them to compare and contrast them, to try to work out how they might be related." It is this "work[ing] out how they might be related" that reflects the essence of extracting.

Different from the earlier stages that involve use, the three types of use in this final stage appear to be hierarchical. That is, we anticipate teachers first developing skill at supporting students in thinking about their peers' ideas, followed by increasing their abilities to create space for students to establish meaning from their peers' thinking, before finally being able to help students to see the underlying mathematics that the student thinking embodies. It is this final use that fully capitalizes on the potential of student thinking to improve the learning of mathematics.

## DISCUSSION AND CONCLUSION

The perceptions and their accompanying uses represent a continuum of less to more productive ways of incorporating student mathematical thinking into instruction. Valuing student participation and student mathematical thinking is important, but on their own they do not make student mathematical thinking available for use in instruction. Likewise eliciting student mathematical thinking is a critical component of PUMT, but when it is thought of as an end in itself—rather than as a means toward building mathematical understanding—it fails to take full advantage of the possibilities student thinking offers. Interpreting student mathematical thinking allows for a broader range of productive use, but in these uses the teacher takes on the mathematical work, thus limiting students' opportunities to engage with the mathematics at a deep level. Building incorporates valuing, eliciting, and interpreting, but uses the information gained from interpreting the student mathematical thinking to turn that thinking back to the students. The productivity of uses categorized as building increases as one moves beyond asking students to ponder their peers' mathematical thinking, to engaging them in mutual sense making of that thinking in order to establish a mutual understanding, to collectively extracting important underlying mathematical ideas as a result of making the student thinking the object of discussion.

The PUMT HLP provides a starting place for conceptualizing PUMT and demonstrates that such a conceptualization is possible and worthy of additional investigation. The HLP could be further refined through using it to analyse more interviews as well as other sources of data, such as videotapes of classroom practice. The HLP could also prove useful as a means of analysing teachers' instruction to gauge proficiency with respect to this particular practice. We envision this work leading to the development of a HLT that could be used to support teachers in developing PUMT. As a result, this critical practice would gain greater traction both in research and in classrooms.

**Acknowledgement**

This research report is based on work supported by the U.S. National Science Foundation (NSF) under Grant Nos. 1220141, 1220357 and 1220148. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

**References**

Ball, D. L., & Cohen, D. K. (1999). Developing practices, developing practitioners. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3-32). San Francisco, CA: Jossey-Bass.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empsona, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*, 403-34.

Feiman-Nemser, S., & Remillard, J. (1996). Perspectives on learning to teach. In F. Murray (Ed.), *A knowledge base for teacher educators* (pp. 63-91). New York, NY: Jossey-Bass.

Franke, M. L., & Kazemi, E. (2001). Learning to teach mathematics: Focus on student thinking. *Theory Into Practice*, *40*, 102-109.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*, 372-400.

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., . . . Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education*, *64*, 226-243.

Leatham, K. R. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education*, *9*, 91-102. doi: 10.1007/s10857-006-9006-8

Leatham, K. R. (2009). *Comments on "Teachers' perceptions about the purpose of student teaching*." Retrieved from http://mathedresearch.blogspot.no/2009/10/teachers-perceptions-about-purpose-of.html

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2014). *Conceptualizing mathematically significant pedagogical opportunities to build on student thinking*. Manuscript submitted for publication.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2007). *Mathematics teaching today: Improving practice, improving student learning* (2nd ed.). Reston, VA: Author.

Peterson, B. E., & Leatham, K. R. (2009). Learning to use students' mathematical thinking to orchestrate a class discussion. In L. Knott (Ed.), *The role of mathematics discourse in producing leaders of discourse* (pp. 99-128). Charlotte, NC: Information Age Publishing.

Remillard, J. T. (1999). Curriculum materials in mathematics education reform: A framework for examining teachers' curriculum development. *Curriculum Inquiry*, *29*, 315-342.

Sherin, M. G. (2002). A balancing act: Developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education*, *5*, 205-233.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*, 114-145.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation*, *2*(1), 50-80.

Van Zoest, L. R., Stockero, S. L., & Kratky, J. L. (2010). Beginning mathematics teachers' purposes for making student thinking public. *Research in Mathematics Education*, *12*, 37-52.