MATHEMATICALLY IMPORTANT PEDAGOGICAL OPPORTUNITIES

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The mathematics education community values using student thinking to develop mathematical concepts, but the nuances of this practice are not clearly understood. For example, not all student thinking provides the basis of productive discussions. In this paper we describe a conceptualization of instances in a classroom lesson that provide the teacher with opportunities to extend or change the nature of students' mathematical understanding—what we refer to as Mathematically Important Pedagogical Opportunities (MIPOs). We analyze classroom dialogue to illustrate how this lens can be used to make more tangible the often abstract but fundamental goal of pursuing students' mathematical thinking.

Research in mathematics teacher education suggests the benefits of instruction that builds on student thinking (e.g., Fennema et al., 1996), but such instruction is complex and difficult both to learn and to enact (Ball & Cohen, 1999; Feiman-Nemser, 2001; Sherin, 2002). Often opportunities to use student thinking to further mathematical understanding either go unnoticed or are not acted upon by teachers, particularly novices (Peterson & Leatham, 2009; Stockero, Van Zoest, & Taylor, 2010). Despite a growing number of teachers who are convinced of the value of student thinking and the need to encourage it, neither teachers nor those who educate them have a clear understanding of how that thinking can best be used to develop mathematical concepts (Peterson & Leatham, 2009; Van Zoest, Stockero, & Kratky, 2010). We address this issue by providing a conceptual framework for thinking about the mathematically important pedagogical opportunities provided by student thinking.

Although skilled teachers and teacher educators often intuitively "know" when important mathematical moments occur during a lesson and can readily produce ideas about how to capitalize on them, the literature reveals a construct that is not well-defined. Ideas related to these instances are mentioned in many different ways. For example, Jaworski (1994) refers to such opportunities as "critical moments in the classroom when students created a moment of choice or opportunity" (p. 527). Davies and Walker (2005) use the term "significant mathematical instances" (p. 275) and Davis (1997) calls them "potentially powerful learning opportunities" (p. 360). Schoenfeld (2008) refers to such moments as "the fodder for a content-related conversation" (p. 57), as "an issue that the teacher judges to be a candidate for classroom discussion" (p. 65) and as the "grist for later discussion or reflection" (p. 70). Schifter (1996) spoke of "novel student idea[s] that prompt teachers to reflect on and rethink their instruction" (p. 130).

It is clear from this literature that these instances, whatever they are called, are important to mathematics teaching and learning. It is also clear that there are some critical mathematical and pedagogical characteristics of such moments. In particular, references to them allude to important mathematics, pedagogical opportunities, and student thinking. We consider these three criteria and focus on their intersection as being the location of Mathematically Important Pedagogical Opportunities (MIPOs). A better understanding of the MIPO construct can inform the work of facilitating and researching teachers' use of students' thinking in mathematically productive ways. One difficulty in learning to use students' mathematical thinking is that there are so many different ways it can interpreted and acted upon. As Lewis (2008) observed, "The 'real' classroom experience is elusive: each moment is experienced differently by the actors involved and their perceptions of those experiences change with time and reflection. The choices of what to focus on, which story to follow, are endless" (p. 5). One critical role that mathematics teacher education can play is to provide lenses, informed by research and advocated by the community at large, for teachers to use both as they teach and as they reflect on and learn from their teaching. The conceptual framework put forth in this paper is designed to be such a lens. In the following, we carefully define and describe the MIPO construct, and initiate a discussion about how teachers and teacher educators might profitably use it to support students' mathematical learning.

Defining Mathematically Important Pedagogical Opportunities

We define a Mathematically Important Pedagogical Opportunity (MIPO) as an instance in a classroom lesson that provides the teacher with an opportunity to extend or change the nature of students' mathematical understanding. To be considered a MIPO, an instance needs to meet two important criteria: it needs to involve *important mathematics* and be a *pedagogical opportunity*.

Mathematically Important

To be *mathematically important* in a given classroom, the instance must be centered on an idea related to mathematical goals for student learning. In the narrowest sense, this would be a mathematical goal for the lesson in which it occurs, but more broadly, it could also be related to the goals for a unit of instruction, an entire course, or for understanding mathematics as a whole (see Figure 1). In the first case, the instance may focus on a particular mathematical idea or connections among ideas within the lesson, while in the latter cases, the instance might involve making connections to other areas of mathematics or developing mathematical ways of thinking.

Figure 1. *Layers of mathematical goals to which a MIPO may relate.*

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There appears to be an inverse relationship between the distance of the underlying mathematical idea in the instance from the goals of the day's lesson and the needed power of the mathematical idea for it to meet the mathematically important requirement of a MIPO. That is, the threshold for mathematical importance increases the further one moves away from the center in Figure 1. Furthermore, the mathematical importance of an instance is relative to the context in which it occurs. For example, an instance considered mathematically important in a calculus class because it highlights a subtlety in the topic being studied would not qualify as mathematically important in the context of an introductory algebra course if the students did not have the background knowledge to make sense of it. On the other hand, an instance that highlights something crucial about the nature of mathematics could qualify as mathematically important in any classroom in which it was accessible to the students and would help them to better understand mathematics as a whole.

When considering whether an instance is mathematically important, it is necessary to distinguish between its relationship to mathematical goals and to other goals for student learning. For example, helping students work more productively in small groups is a goal in many classrooms. Although this goal may support students' mathematical learning, the goal itself is not mathematical in nature, and thus, an instance related only to this goal would not be a MIPO—a MIPO must be firmly grounded in important mathematics.

Pedagogical Opportunity

In addition to involving important mathematics, a MIPO requires a *pedagogical opportunity*. Pedagogical opportunities are observable student actions that provide an opening for working towards an instructional goal. As such, pedagogical *opportunities* can be cultivated by the teacher, but cannot be created independently of the students. Teachers routinely make pedagogical *moves* that are designed to create opportunities for students to learn mathematics, such as posing quality tasks, asking probing questions, assessing students' progress and modifying their instruction in response to additional information. Well-executed pedagogical moves can, in fact, increase the likelihood of pedagogical opportunities in a teacher's class, but the opportunities themselves come from the students, not the teacher. For example, if a teacher were to introduce a theoretical student error into the class discussion to help clarify an issue that she felt her students were struggling with, it would remain a pedagogical move and not an opportunity until a student or students in the class interacted with the error publicly. A pedagogical opportunity must be grounded in an observable student action. Student actions, by providing insight into student engagement with an instructional goal, provide an opening for the teacher to work toward achieving that goal.

The Intersection of Mathematically Important and Pedagogical Opportunity

Mathematically Important Pedagogical Opportunities (MIPOs) occur at the intersection of *important mathematics* and *pedagogical opportunities*. In this intersection, observable student actions provide pedagogical openings for working towards mathematical goals for student learning. Although simple yes/no answers or other utterances may provide evidence *that* students are thinking, these openings only occur when student actions provide insight into *what* students are thinking about mathematical ideas. Thus, *observable student thinking* underlies the MIPO construct. Our conception of the relationship among important mathematics, pedagogical opportunities and student thinking is shown Figure 2. In this section we elaborate on each region in Figure 2 to further clarify the MIPO construct.

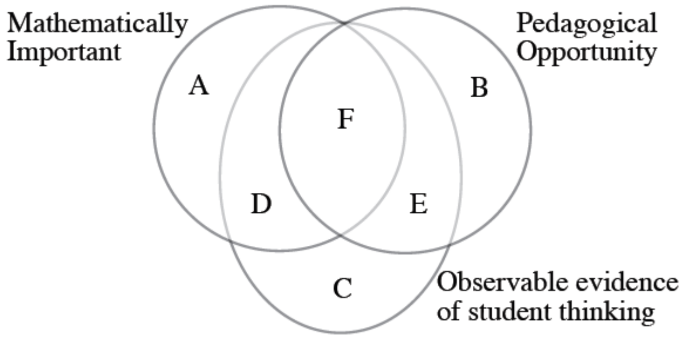

Figure 2. *Relationship among important mathematics, pedagogical opportunities and student thinking.*

Region A represents situations that are mathematically important, but neither provide evidence of student thinking nor a pedagogical opportunity. A teacher presenting important mathematical information would fall into this region, as would situations in which a teacher makes a pedagogical move to engage students with the mathematics, but students fail to provide observable evidence of having done so. In general, these are situations where important mathematics is present, but observable evidence of student thinking is not. For example, if a teacher were to make a mistake on the board related to important mathematics in the lesson, it would be a mathematically important moment. If the teacher corrects the error and moves on without student engagement with the error, this moment would not provide an opportunity to extend or change the nature of students' mathematical understanding. Student actions in response to the error that reveal their mathematical thinking, such as asking questions that illuminate the key mathematics behind the error, would provide an opening for working towards an instructional goal. This would put the moment in the intersection of the three areas in Figure 2, thus classifying it as a MIPO.

Region B represents situations where student actions do not provide evidence of student thinking and are not mathematically important, yet provide inroads for important pedagogical goals. For example, if a student were to get up to sharpen his pencil in the middle of a class discussion, the action would not provide insight into his thinking, but it could provide a pedagogical opportunity to talk about important classroom norms. In fact, pedagogical opportunities that neither provide evidence of student thinking nor relate to important mathematics seem to relate to general pedagogical rather than content-specific goals.

Region C represents student actions that provide evidence of their thinking, but the thinking is neither about important mathematics nor related to instructional goals. For example, a student might make a comment about the length of a homework assignment or reiterate a memorized fact. Although these comments give the teacher information about the student's thinking, they neither connect to important mathematics nor provide an opening for working towards an instructional goal, and thus, do not meet the criteria for a MIPO.

Region D represents situations that involve important mathematics and evidence of student thinking, but do not provide an opening for working towards an instructional goal. For example, a student in an algebra class could eloquently summarize why adding a constant to a linear equation corresponds to a vertical shift of the graph. While this comment could make a positive contribution by summarizing what students already know, it would not create the opportunity to extend or change the nature of students' mathematical understanding.

Region E represents pedagogical opportunities that provide insight into student thinking, but are not related to important mathematics. For example, a student might say, “I don’t see why I need to think by myself for one minute before I talk with my group.” This is not related to mathematics, but it does provide observable evidence of student thinking and would provide an opening for discussing the instructional goal of allowing individuals time to formulate their own thoughts before being influenced by others.

Region F represents situations in which student thinking about an important mathematical idea provides an opening for working towards a mathematical goal for student learning. This is what creates a MIPO. In this region, a student might, for instance, question or comment on a mathematical idea, verbalize their incomplete thoughts as they try to make sense of a mathematical idea, express incorrect mathematical thinking, make an error, or notice a mathematical contradiction. In all these instances, what is important is that the student thinking provides an opening for the teacher to make a pedagogical move that will extend or change the nature of students’ understanding of important mathematics.

Because student thinking is at the heart of every MIPO, there is no region in Figure 2 that includes both important mathematics and pedagogical opportunity without involving observable evidence of student thinking. It is when student thinking is made public that teachers have an opportunity to use that thinking to further students’ mathematical understanding. Although MIPOs can occur in any classroom environment, they are more likely to occur in classrooms that provide ample opportunity for students to make their thinking public.

An Example from the Literature

In this section we illustrate the MIPO construct by using it as a lens to analyze a piece of transcript taken from Leinhardt and Steele (2005, pp. 107-108). The episode comes from a 5th grade class discussion about finding output values for the rule 3x + 1 given different input values. The teacher, Magdalene Lampert, added ¼ as an input value to a table of input-output values and asked for the output. All of the previous input values had been whole numbers.

Soochow: One and three fourths.
T: How would you explain it please?
Soochow: *Because one-fourth times three is three-fourths and then you just add o– add a one.*
T: Okay, so first you times by three and then you add one.
T: Who can explain why one fourth times three is three fourths? Sun Wu?
Sun Wu: *One fourth, like one fourth of a pie and then somebody brings two more and one times three is three—three pieces of pie that came out of four pieces of pie?*
T: Okay, are they all the same size? Those three pieces of pie? Lisa?
Lisa: Yes
T: How do you know?
Lisa: Because if you’re adding one fourth times three you’re going to [—] [—] equal parts
T: Okay. Cause I’m, I’m taking three things that are all the same size. They’re all the size of one fourth. Ali?
Ali: *It could be one fourth [—] could be a whole one.*

T: Can you explain what you mean?

Ali: Can I come to the board?

T: Yes, here take this, [chalk] it's easier to see.

Ali: Here's like a big pie [draws circle and divides it into fourths]

T: Um-hum.

Ali: And then you could divide it into fourths, four pieces. *And then one fourth could be one (points to one segment of circle) and then would be like this one (points to the 1 on the input side of the chart).*

T: I don't understand what you mean. Does anybody else understand what Ali means? Bridgette?

Bridgette: Me-, he means that if you ha-, if you have one fourth and you make say you color in three of the four pieces [—] equal one whole.

T: Is that what you meant?

Bridgette: Yeah.

T: Okay, what do you think about that? Ali is saying three times one fourth is one fourth [*sic*]. Add one fourth and you'd get four so it would be just like here [points to the 4 beside the 1 in the function chart]. But the input number here was one [writes faint 1 in input column beside the 4] and now the input number here is one fourth [points to the ¼ in new chart]. What do you think Sun Wu?

Sun Wu: *He thinks the um, the one is like one fourth. But it's really one, another, four.*

T: What do you think about that Ali? [draws another circle]. How many fourths are there in one whole?

Ali: Four fourths [T draws new circle divided into fourths].

T: Four fourths? So if I was going to put a number in here I could put one and a fourth [*sic*] [writes in column]

T: Is there anything I could put in there besides one and a fourth? Elsie?

Elsie: Wouldn't it be one and three fourths?

T: Oh, I'm sorry. It should be one and three fourths like that anyway [changes chart]. Is that what you meant?

Elsie: Yeah.

We now use Figure 2 to analyze excerpts from this dialogue. In doing so we highlight examples from each region of the figure in order to help the reader distinguish instances that are MIPOs from those that are not. Lines 17 and 22 are examples of Region A, as in both cases the teacher emphasizes the same size of the pieces—an important mathematical principle. There is no student thinking involved and no pedagogical opportunity, but it is mathematically important. The student's inquiry in Line 26 about coming to the board provides an opportunity to address the teacher's expectations for sharing one's work and using tools in the class, but neither provides insight in the student's thinking nor involves important mathematics; thus, it falls in Region B. Lines 7, 24, 28, 36 and 51 provide evidence of student thinking, but do not involve important mathematics or provide pedagogical opportunities, thus they are examples of Region

C. Note that this dialogue also contains several student utterances that fell short of being observable evidence of student thinking (Lines 18, 40 & 59). Although these utterances suggest that the students are thinking, they don't provide insights into what they are thinking. In Line 56 a student provides a correction to an error made by the teacher. This is evidence of the student's thinking and also involves important mathematics, but does not provide a pedagogical opportunity, thus it falls in Region D. In Line 20, we see evidence of a student's thinking, but it isn't clear exactly what is going on mathematically. Because of this it isn't possible to determine if it involves an important mathematical issue or if it is merely a misspeaking. It does, however, provide a pedagogical opportunity to discuss the importance of the words that we use, thus it falls in Region E. Lines 9, 14, 30, and 47 all deal with observable student thinking, important mathematics, and pedagogical opportunities, thus fall in Region F. In Line 9, for example, Soochow's explanation of how to find the output value provided an opportunity to review how to multiply a fraction by a whole number. Sun Wu's explanation at Line 14 shifted the unit from the whole pie to a piece of the pie, providing a pedagogical opportunity for the teacher to engage her students in further discussion of this important mathematical idea. In each of these cases, the teacher expertly incorporated the MIPO into her instruction and it is possible to see how the discussion supported students in learning about important mathematics.

Discussion and Conclusion

Researchers and practitioners in mathematics teacher education advocate the use of student thinking as a means of improving mathematics instruction. Many teachers we have observed, particularly novices, seem to interpret this call to mean that *all* student thinking is equally valuable and, thus, should *all* be pursued in similar ways. We argue, however, that this is not the case. While teachers certainly need to carefully listen to all student ideas, this listening must be followed by thoughtful consideration of whether a particular idea or comment is worth pursuing.

By highlighting three critical components of instances in a classroom that provide opportunities to advance students' mathematical understanding—important mathematics, pedagogical opportunity and student thinking—the MIPO construct can be used as a tool to help teachers learn to distinguish moments that provide opportunities to further students' mathematical learning from those that do not. In addition, it provides a tool for helping teachers make sense of classroom situations—what Levin, Hammer and Coffey (2009) call *framing*. From this perspective, whether a teacher notices the value in an event depends on how he or she frames what is taking place during instruction. If, for example, a teacher views a student error as something that needs to be corrected, he or she is unlikely to consider the mathematical thinking behind the error or whether the error could be used to highlight a specific mathematical idea. On the other hand, a teacher who views an error as a site for learning is more likely to consider both the mathematics underlying the error and how it could be used to develop mathematical understanding. In considering whether an instance is a MIPO, teachers need to frame instances of student thinking in terms of both mathematical importance and the pedagogical opportunity they provide. Framing classroom events in this way has the potential to change the way teachers analyze and act upon instances in the classroom.

The conceptualization of MIPOs we have described provides both teachers and teacher educators a lens for analyzing the complexity of classroom mathematics discourse and a vocabulary for discussing the mathematical and pedagogical importance therein. We believe such a lens is significant because MIPOs are high-leverage instances of student thinking that have the potential to change the nature of mathematics instruction if incorporated well into a lesson. This conceptualization of MIPOs provides a tool that can help make more tangible the often abstract but fundamental goal of building on students' mathematical thinking.

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