Engaging Teachers in Identifying the Point of Student Mathematical Thinking

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Plan for the session

- Background – Student Mathematical Thinking
- Launch – *An Instance of Student Thinking*
- Explore – Mathematical Point Card Sort
- Summary – Define Mathematical Point
- Discussion – Implications for Teacher Education

Student Mathematical Thinking

- Not all student mathematical thinking is the same–MOSTs (Leatham, Peterson, Stockero & Van Zoest, 2015)

Mathematically significant
pedagogical **Opportunities**
to build on **in Student**
Thinking

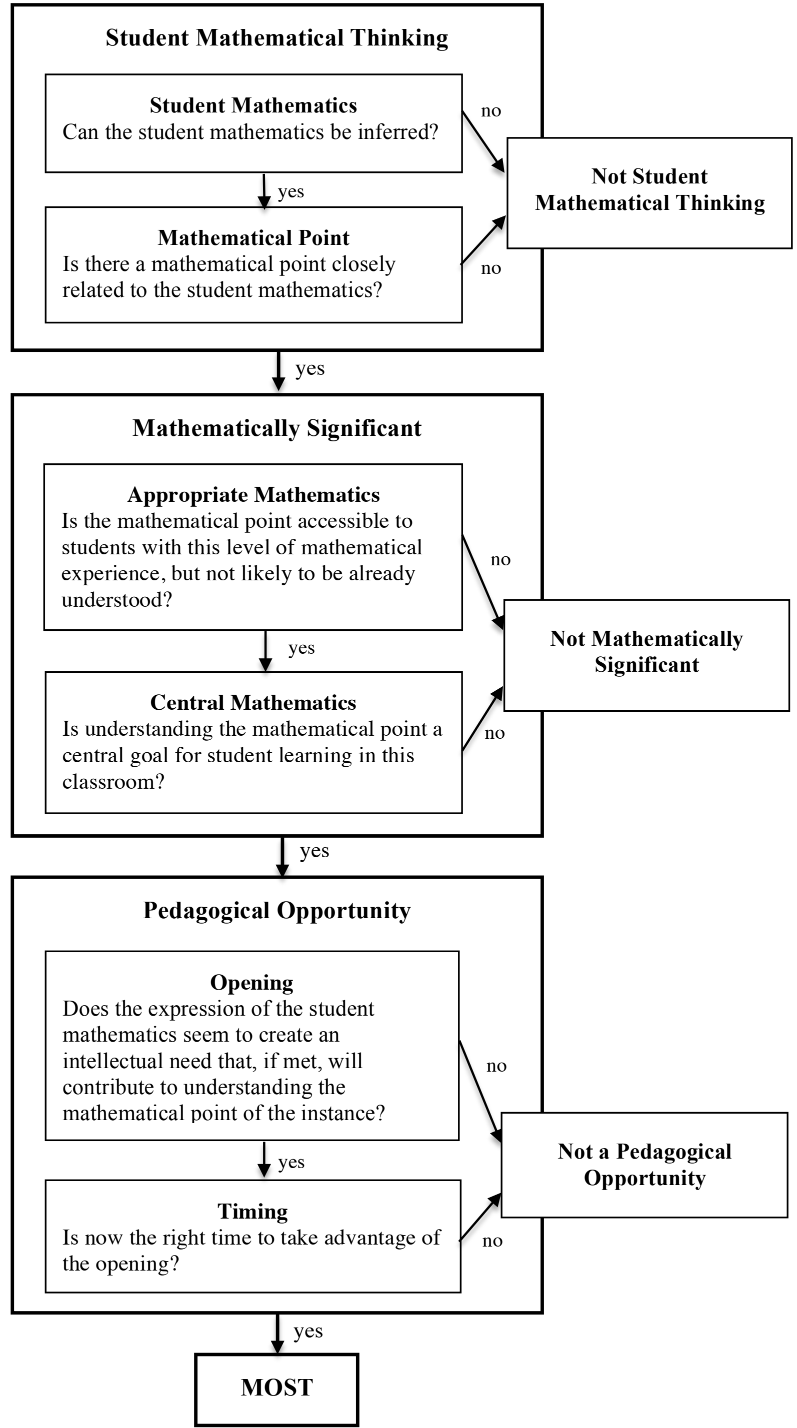
M O
S T

MOST
Analytic Framework

Stockero, S. L., Peterson, B. E., Leatham, K. R., & Van Zoest, L. R. (2014). The "MOST" productive student mathematical thinking. *Mathematics Teacher,* 108(4), 308-312.

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education,* 46(1), 88-124.

LeveragingMOSTS.org

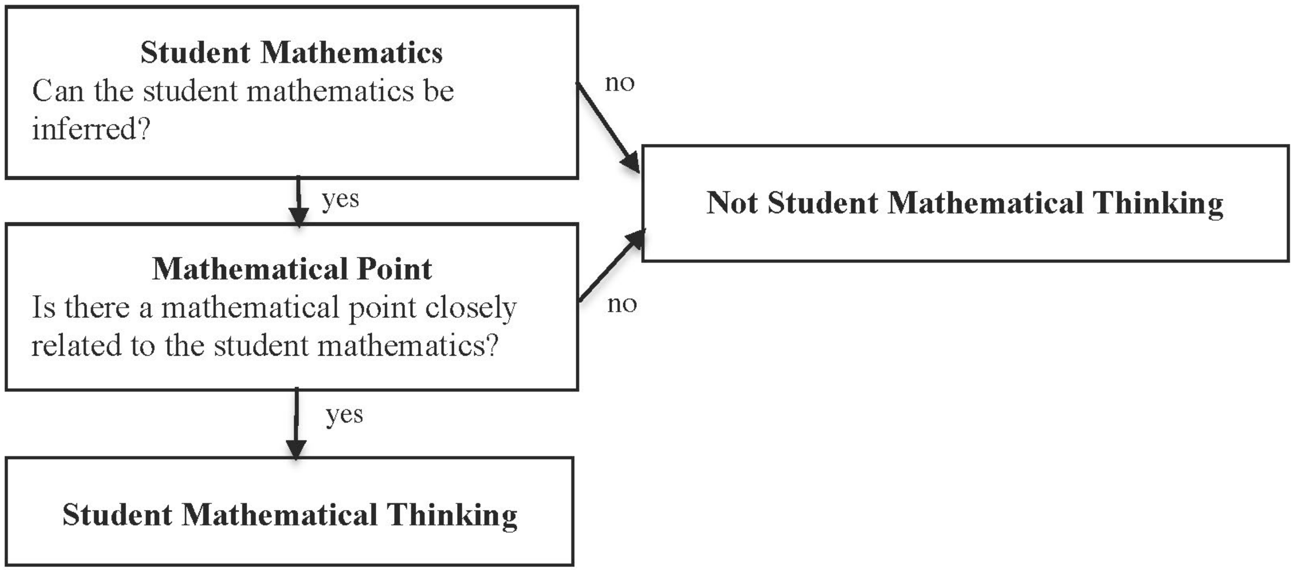


MOSTs are opportunities…

…for the teacher to make student thinking the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.

…to **build** on student thinking toward the **mathematical point**.

Student Mathematical Thinking



Student Mathematical Thinking

- Not all student mathematical thinking is the same–MOSTs (Leatham, Peterson, Stockero & Van Zoest, 2015)
- Mathematical Point
  - necessary to determine whether an instance of student thinking is a MOST
  - what the MOST can be used to build towards

An Instance of Student Thinking

A teacher in an 8th grade algebra class that had been working on writing linear equations from a table of values for a specific situation asked, "How do we find the slope given any table?" and put this (to the right) generic table of values on the board for the students to use in their explanation. A student replied, "I found the slope by subtracting the y-values in the table, 21 - 19, so the slope is 2."

X	Y
0	15
2	19
3	21
5	25

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What mathematical understanding do you think students could gain by considering this instance of student thinking?

Mathematical Understandings
reported out during session

Going from procedure to proportional reasoning

Slope is rate of change

Linear function has a constant slope

Not all functions are linear

What do you do with the data in the table more generally

Finding patterns in the table, when pattern holds true

Relationships between x and y in the table

A linear function has a constant rate of change

Difference between linear functions and proportional relationships

Discussed differences in grainsizes and need for a common language

Mathematical Point Card Sort

Sort the cards into these three categories:

Well-specified statement of a mathematical truth		
Can be gained from considering the SM	Cannot be gained from considering the SM	Not Well-Specified

Mathematical Point Examples and Non-Examples

Context	Student Mathematics (SM)	Well-Specified Statement of a Mathematical Truth		Not Well-Specified
		Could be gained from considering SM	Could not be gained from considering SM	
A. In a beginning algebra lesson on solving simple linear equations, a student says, “To get *m* alone on the left side of the equation $m - 12 = 5$, you can subtract 12.”	To get *m* alone on the left side of the equation $m - 12 = 5$, you can subtract 12.			
B. In a beginning algebra lesson on representing linear situations with equations, the equation (t*2.5)+25 = m is on the board. A student says, “you don’t need the parentheses.”	In the equation (t*2.5)+25 = m, the parentheses around t*2.5 are optional.			
C. In an introductory lesson on adding fractions with like denominators, a student writes 2/5 + 1/5 = 3/10 on the board.	2/5 + 1/5 = 3/10.			
D. On day two of a unit on solving simple linear equations, the teacher writes x = 3 as the solution to $x + 2 = 5$, and a student remarks, “Hey, wait a minute, yesterday you said *x* equals two!”	Yesterday *x* equaled 2 and today *x* equals 3.			

Mathematical Point Examples and Non-Examples

Context	Student Mathematics (SM)	Well-Specified Statement of a Mathematical Truth			Not Well-Specified
		Could be gained from considering SM		Could not be gained from considering SM	
		Mathematical Point	Not Closest to SM		
In a beginning algebra lesson on solving simple linear equations, a student says, "To get m alone on the left side of the equation $m - 12 = 5$, you can subtract 12."	To get m alone on the left side of the equation $m - 12 = 5$, you can subtract 12.	Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation. [A1]	Adding a number and subtracting that same number are inverse operations. [A2]	An integer and its opposite are the same distance from zero on the number line. (Charles, 2005, p. 18) [A3]	Solving linear equations. [A4]
In a beginning algebra lesson on representing linear situations with equations, the equation $(t*2.5)+25 = m$ is on the board. A student says, "you don't need the parentheses."	In the equation $(t*2.5)+25 = m$, the parentheses around $t*2.5$ are optional.	Parentheses are necessary when the intended order of operations differs from the conventional order of operations. [B2]	The commutative property applies to addition and multiplication but not subtraction and division. (Charles, 2005, p. 16) [B4]	The nature of the quantities in a relationship determines what values of the input and output quantities are reasonable. (Charles, 2005, p. 18) [B3]	Order of operations. [B1]
In an introductory lesson on adding fractions with like denominators, a student writes $2/5 + 1/5 = 3/10$ on the board.	$2/5 + 1/5 = 3/10$.	Adding fractional pieces of the same size changes the number of pieces, but not the size of the pieces. [C3]	Adding two quantities means combining the amounts together. [C4]	Every fraction/ratio can be represented by an infinite set of different but equivalent fractions/ratios. (Charles, 2005, p. 18) [C1]	How to get a common denominator when adding fractions. [C2]
On day two of a unit on solving simple linear equations, the teacher writes $x = 3$ as the solution to $x + 2 = 5$, and a student remarks, "Hey, wait a minute, yesterday you said x equals two!"	Yesterday x equaled 2 and today x equals 3.	A letter can be used to represent an unknown quantity in an equation and can represent different quantities for different equations. [D4]	Letters can be used to represent unknown quantities, varying quantities, and arguments for a function. [D1]	Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation. [D2]	The meaning of variable. [D3]

Mathematical Point (MP)

The mathematical understanding (MU)* that

1. students could gain from considering a particular instance of student thinking
2. is most closely related to the student mathematics of the thinking

*well-specified statement of a mathematical truth

Mathematical Point Examples and Non-Examples

Mathematical Point ~ The mathematical understanding (well-specified statement of a mathematical truth) that (1) students could gain from considering a particular instance of student thinking; and (2) is most closely related to the student mathematics of the thinking.

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In a beginning algebra lesson on representing linear situations with equations, the equation (t*2.5)+25 = m is on the board. A student says, "you don't need the parentheses."	In the equation (t*2.5)+25 = m, the parentheses around t*2.5 are optional.	Parentheses are necessary when the intended order of operations differs from the conventional order of operations.	The commutative property applies to addition and multiplication but not subtraction and division. (Charles, 2005, p. 16)	The nature of the quantities in a relationship determines what values of the input and output quantities are reasonable. (Charles, 2005, p. 18)	Order of operations.
In an introductory lesson on adding fractions with like denominators, a student writes 2/5 + 1/5 = 3/10 on the board.	2/5 + 1/5 = 3/10.	Adding fractional pieces of the same size changes the number of pieces, but not the size of the pieces.	Adding two quantities means combining the amounts together.	Every fraction/ratio can be represented by an infinite set of different but equivalent fractions/ratios. (Charles, 2005, p. 18)	How to get a common denominator when adding fractions.
On day two of a unit on solving simple linear equations, the teacher writes x = 3 as the solution to *x* + 2 = 5, and a student remarks, "Hey, wait a minute, yesterday you said *x* equals two!"	Yesterday *x* equaled 2 and today *x* equals 3.	A letter can be used to represent an unknown quantity in an equation and can represent different quantities for different equations.	Letters can be used to represent unknown quantities, varying quantities, and arguments for a function.	Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation.	The meaning of variable.

Why Mathematical Point?

- Instructional practices that use student thinking lead to improved learning. (NCTM, 2014)
- Identifying the MP in student thinking supports effective decision making about which student thinking to use. (MOST Analytic Framework)
- Having the MP in mind supports facilitating productive discussion of the student thinking by the class. (Building)

Some things to keep in mind

- MPs must be accurate, but the student thinking may or may not be.
- An MP exists only in relation to a specific instance of student thinking.
- To be an MP, the mathematical understanding must be gained from considering the student thinking itself.
- Not all instances of student thinking give rise to an MP.
- There are acceptable variations in the articulation of SMs, mathematical understandings, and MPs.

Moving Towards Productive Use of Student Thinking

Ted: “[Mathematical points] allow you to use where the student’s understanding currently is in a way to move them toward the understanding of a mathematical idea you want them to be at. This allows you to build off student ideas yourself or preferably through student discussions in such a way that raises everyone’s understanding of the mathematics being discussed.”

Isabelle: “It is important to have mathematical points in mind because this focuses and guides the lesson towards highlighting important math concepts. It requires understanding of student thinking and interpreting it in order to probe student thinking in the direction toward the point.”

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Possible Mathematical Points:

1. Finding the slope.
2. The slope of a linear equation is $(y_2-y_1)/(x_2-x_1)$.
3. The slope of an equation at a point is the slope of the tangent line.

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Mathematical Point	Not Gained from SM	Not Well-Specified
The slope of a linear equation is $(y_2-y_1)/(x_2-x_1)$.	The slope of an equation at a point is the slope of the tangent line.	Finding the slope.

Discussion

- What is the potential of identifying mathematical points for instances of student mathematical thinking as a “learning to teach” activity for preservice teachers or an “improving teaching” activity for practicing teachers?
- How might articulating a well-specified statement of a mathematical idea closely related to an instance of student thinking require one to unpack their own mathematical understanding and make connections among and within mathematical concepts?

Thanks for coming!

For more information about our work:

Leveraging MOSTS.org