



Productive Use of Student Mathematical Thinking is More than a Single Move

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Leveraging MOSTs: Developing a Theory of Productive Use of Student Mathematical Thinking



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Mathematical
Opportunities
in Student
Thinking

Mathematically significant
pedagogical **O**pportunities
to build on **S**tudent
Thinking





Mathematical
Opportunities
in Student
Thinking

Mathematical Opportunities in Student Thinking





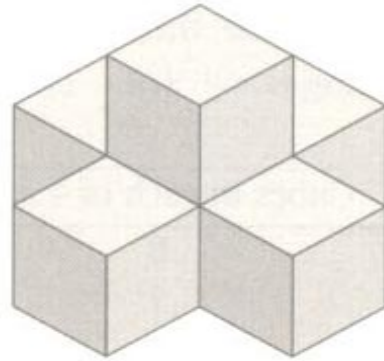
What is meant by “build on”?

- “*build on* student mathematical thinking” (Hill, Ball, & Schilling, 2008, p. 378)
- “*build on* student thinking during instruction” (Van Zoest & Stockero, 2012, p. 43)
- “*build on* and honor student mathematical thinking” (NCTM, 2014, p. 30)
- “*build on* student thinking and also advance important mathematical ideas” (Stein, Engle, Smith, & Hughes, 2008, p. 314)
- “whole class discussions that *build on* student thinking and guide the learning of the class” (NCTM, 2014, p. 35)

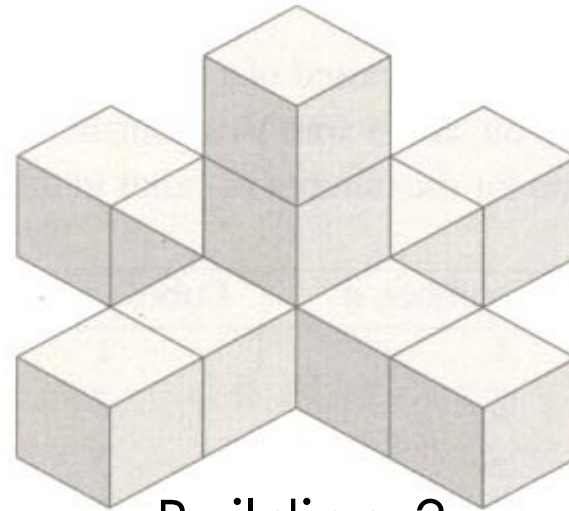
An Example



Building 1



Building 2

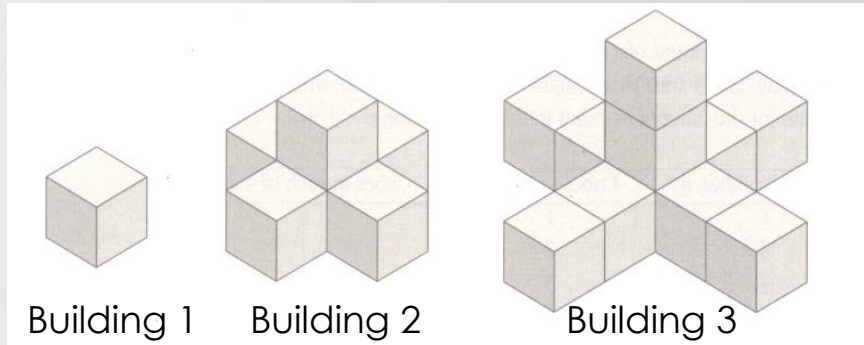


Building 3

Study the sequence of cubes above. Assuming the sequence continues in the same way, how many cubes will there be in the n th building.

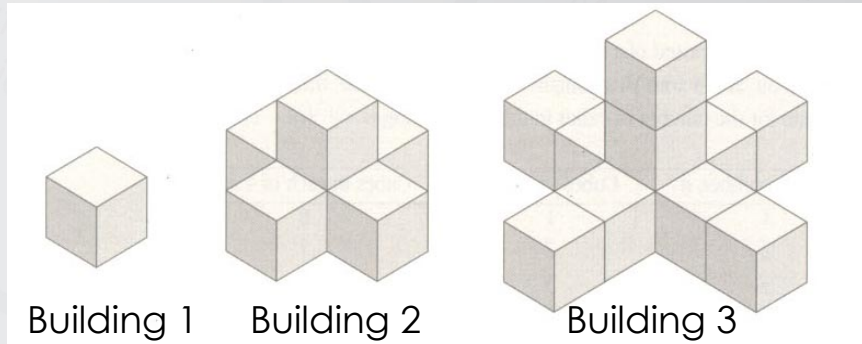
Adapted from "Counting Cubes", Lappan, Fey, Fitzgerald, Friel, & Phillips (2004).
Connected Mathematics - Say it with symbols: Algebraic Reasoning

An Example



Student Solution A: The equation is $5n + 1$ where n equals the length of 1 individual arm excluding the middle cube. You would multiply that by 5, because there are 5 arms, and then add 1 for the middle cube, and that'll give you the number of cubes.

An Example



Student Solution B: We came up with $5n - 4$
There are 5 arms and n is the building number.
Since each arm includes the middle block, we
need to subtract 4 at the end.

Is this Building?



Mathematical
Opportunities
in Student
Thinking

A teacher invites these two students to share their different solutions.

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Is this Building?



Mathematical
Opportunities
in Student
Thinking

A teacher invites these two students to share their different solutions.

Teacher Follow-up: “The expressions are different because the variable is defined differently in each case.”

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Is this Building?



Mathematical
Opportunities
in Student
Thinking

A teacher invites these two students to share their different solutions.

Teacher Follow-up: “Both students have been able to count the number of blocks correctly for the 4th building and the 17th building but they have non-equivalent expressions. How is this possible?”

Building ...



Mathematical
Opportunities
in Student
Thinking

- is a complex practice
- is not a single move
- must be a collection of moves

How would we recognize building if we saw it?

Principles Underlying Productive Use of MOSTs



Mathematical
Opportunities
in Student
Thinking

- The mathematics of the MOST is at the forefront.
- Students are positioned as legitimate mathematical thinkers.
- Students are engaged in sense making.
- Students are working collaboratively.

Our Definition of “Build”



Building on student mathematical thinking occurs when teachers make student mathematical thinking the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.

Analyzing Teacher Practice



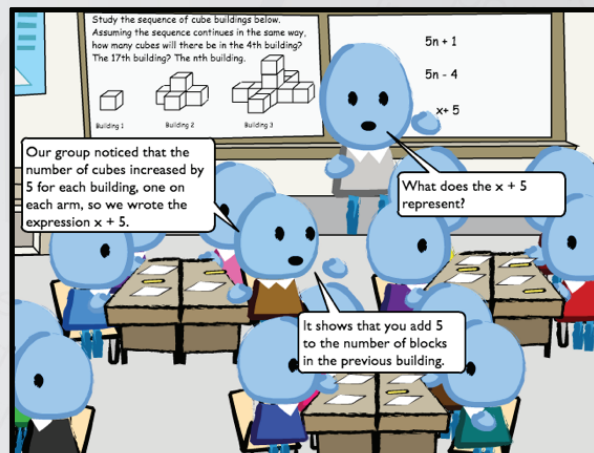
Mathematical
Opportunities
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With this definition of building in mind identify which scenarios seem to be productive (or less productive) ways of building on student thinking. Justify your answer.

Building on student mathematical thinking occurs when teachers make student mathematical thinking the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.

Analyzing Lesson Sketch Episodes

Scenario A



Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The nth building.

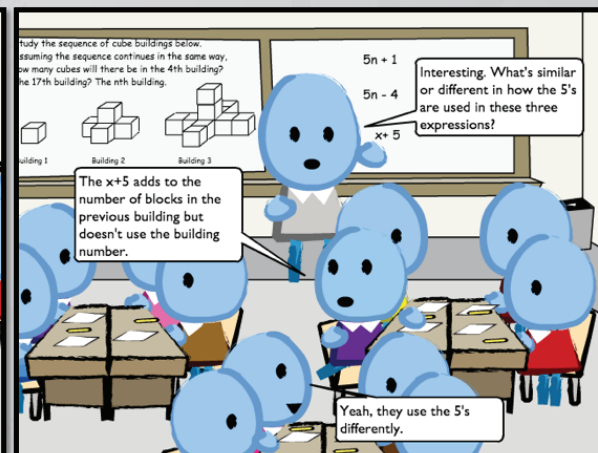
Building 1: 1 cube
Building 2: 4 cubes
Building 3: 9 cubes

$5n + 1$
 $5n - 4$
 $x + 5$

Our group noticed that the number of cubes increased by 5 for each building, one on each arm, so we wrote the expression $x + 5$.

What does the $x + 5$ represent?

It shows that you add 5 to the number of blocks in the previous building.



study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The nth building.

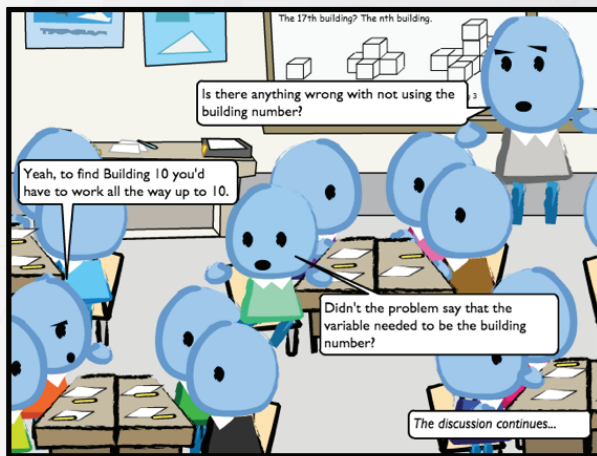
Building 1: 1 cube
Building 2: 4 cubes
Building 3: 9 cubes

$5n + 1$
 $5n - 4$
 $x + 5$

Interesting. What's similar or different in how the 5's are used in these three expressions?

The $x+5$ adds to the number of blocks in the previous building but doesn't use the building number.

Yeah, they use the 5's differently.



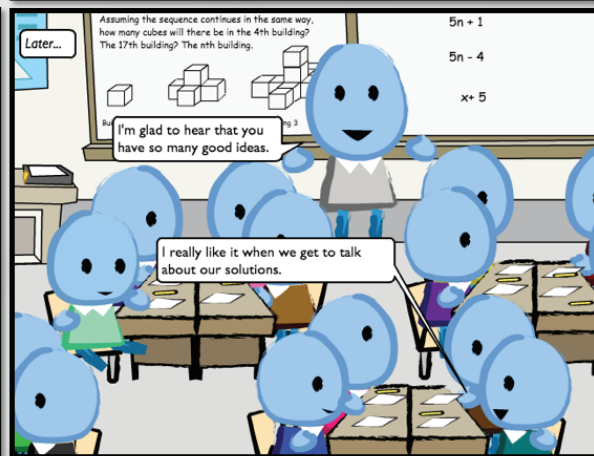
The 17th building? The nth building.

Is there anything wrong with not using the building number?

Yeah, to find Building 10 you'd have to work all the way up to 10.

Didn't the problem say that the variable needed to be the building number?

The discussion continues...



Later...

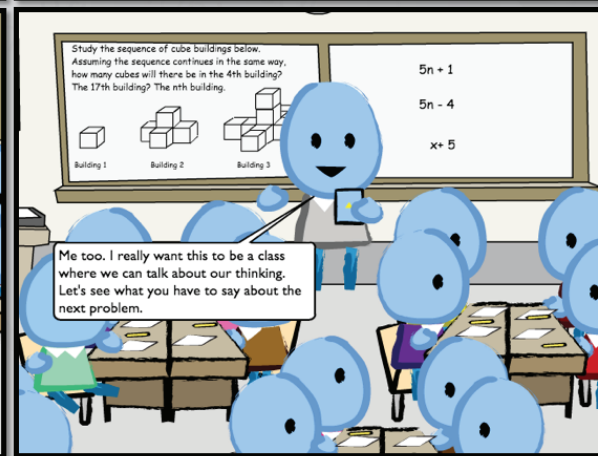
Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The nth building.

Building 1: 1 cube
Building 2: 4 cubes
Building 3: 9 cubes

$5n + 1$
 $5n - 4$
 $x + 5$

I'm glad to hear that you have so many good ideas.

I really like it when we get to talk about our solutions.



Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The nth building.

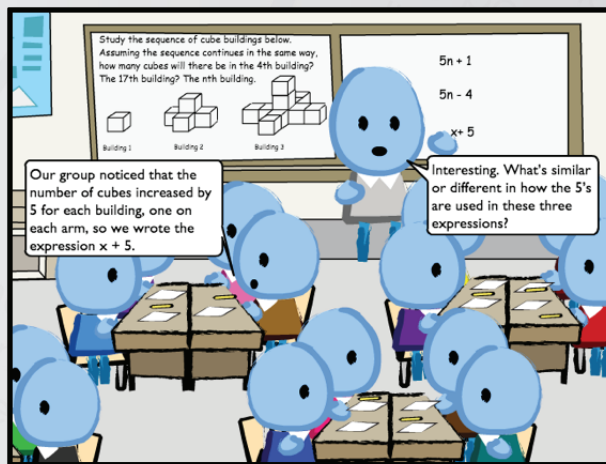
Building 1: 1 cube
Building 2: 4 cubes
Building 3: 9 cubes

$5n + 1$
 $5n - 4$
 $x + 5$

Me too. I really want this to be a class where we can talk about our thinking. Let's see what you have to say about the next problem.

Analyzing Lesson Sketch Episodes

Scenario B



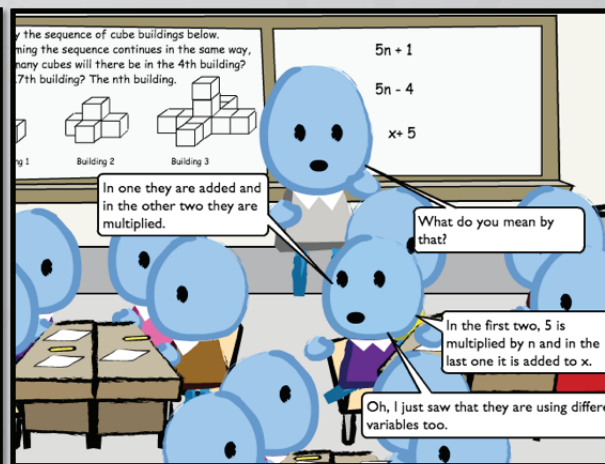
Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The nth building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

Our group noticed that the number of cubes increased by 5 for each building, one on each arm, so we wrote the expression $x + 5$.

Interesting. What's similar or different in how the 5's are used in these three expressions?



By the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 7th building? The nth building.

Building 1 Building 2 Building 3

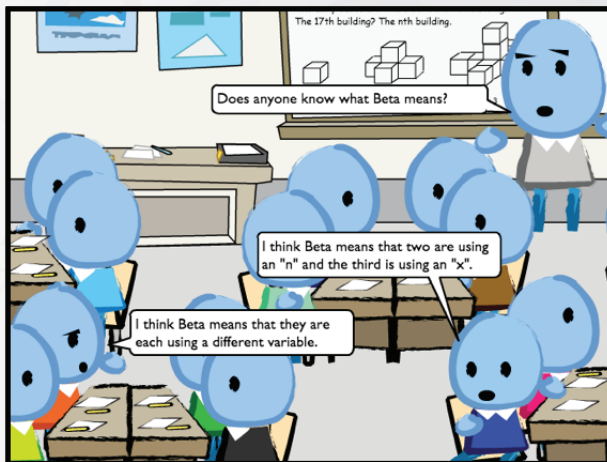
$5n + 1$
 $5n - 4$
 $x + 5$

In one they are added and in the other two they are multiplied.

What do you mean by that?

In the first two, 5 is multiplied by n and in the last one it is added to x.

Oh, I just saw that they are using different variables too.

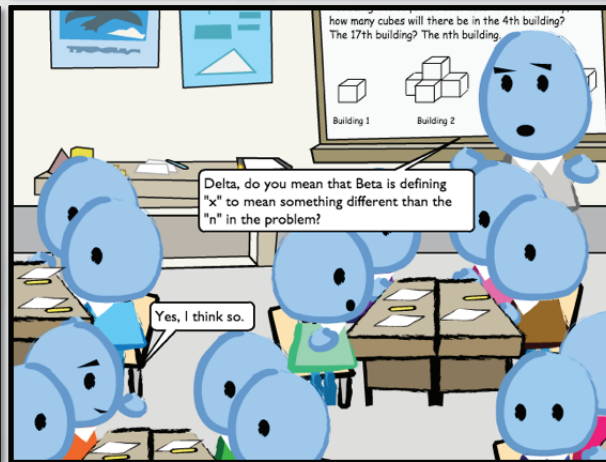


The 17th building? The nth building.

Does anyone know what Beta means?

I think Beta means that two are using an "n" and the third is using an "x".

I think Beta means that they are each using a different variable.

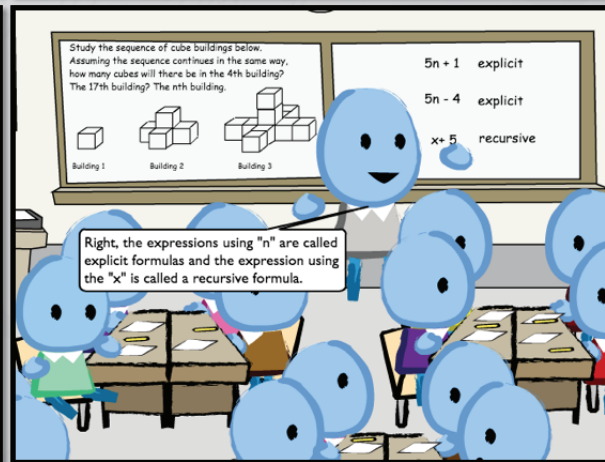


how many cubes will there be in the 4th building? The 17th building? The nth building.

Building 1 Building 2

Delta, do you mean that Beta is defining "x" to mean something different than the "n" in the problem?

Yes, I think so.



Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The nth building.

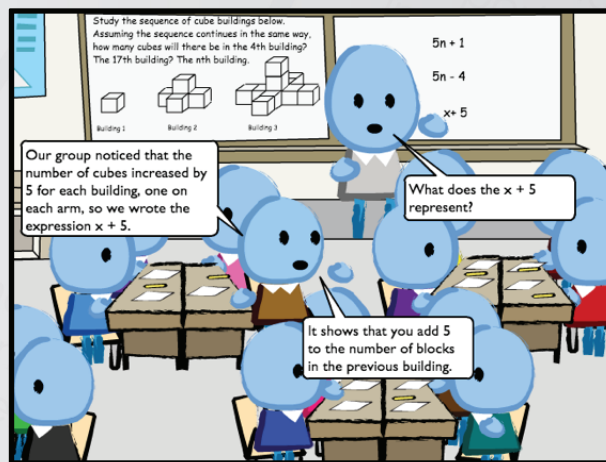
Building 1 Building 2 Building 3

$5n + 1$ explicit
 $5n - 4$ explicit
 $x + 5$ recursive

Right, the expressions using "n" are called explicit formulas and the expression using the "x" is called a recursive formula.

Analyzing Lesson Sketch Episodes

Scenario C



Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

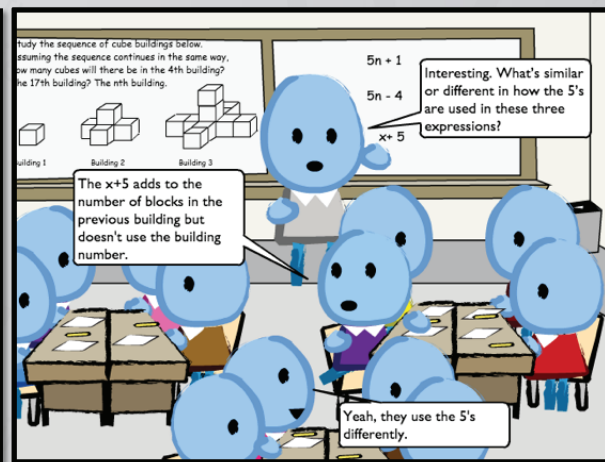
Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

Our group noticed that the number of cubes increased by 5 for each building, one on each arm, so we wrote the expression $x + 5$.

What does the $x + 5$ represent?

It shows that you add 5 to the number of blocks in the previous building.



Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

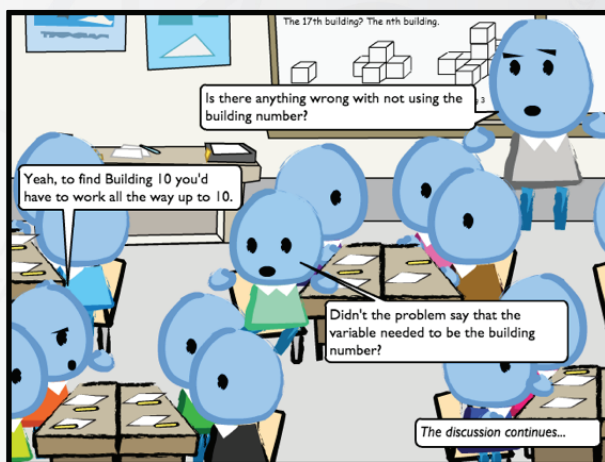
Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

Interesting. What's similar or different in how the 5's are used in these three expressions?

The $x+5$ adds to the number of blocks in the previous building but doesn't use the building number.

Yeah, they use the 5's differently.



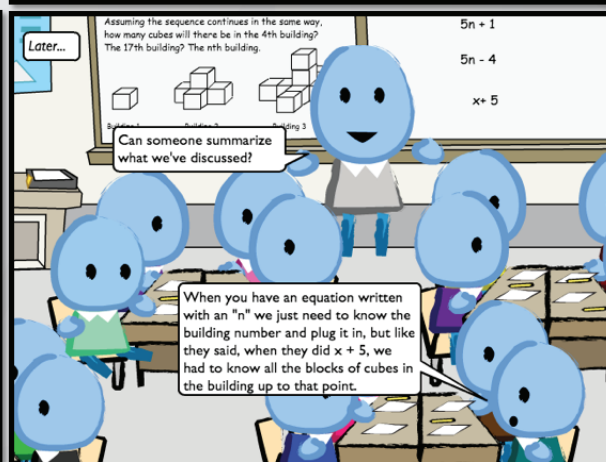
The 17th building? The n th building.

Is there anything wrong with not using the building number?

Yeah, to find Building 10 you'd have to work all the way up to 10.

Didn't the problem say that the variable needed to be the building number?

The discussion continues...



Later...

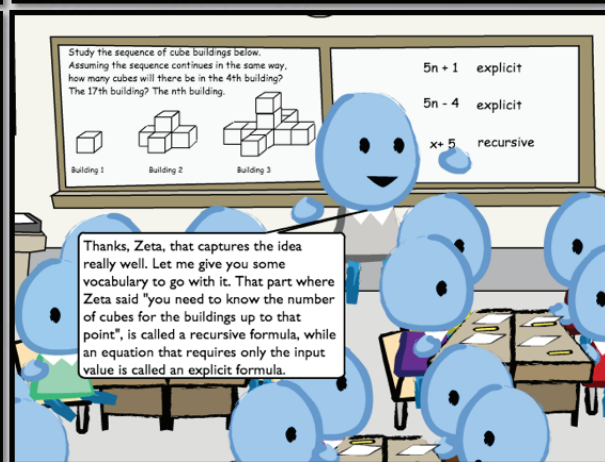
Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

Can someone summarize what we've discussed?

When you have an equation written with an "n" we just need to know the building number and plug it in, but like they said, when they did $x + 5$, we had to know all the blocks of cubes in the building up to that point.



Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$ explicit
 $5n - 4$ explicit
 $x + 5$ recursive

Thanks, Zeta, that captures the idea really well. Let me give you some vocabulary to go with it. That part where Zeta said "you need to know the number of cubes for the buildings up to that point", is called a recursive formula, while an equation that requires only the input value is called an explicit formula.

Analyzing Lesson Sketch Episodes



Mathematical
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Scenario D

Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$

Our group noticed that the number of cubes increased by 5 for each building, one on each arm, so we wrote the expression $x + 5$ to show that you add 5 to the number of blocks in the previous building.

Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

Interesting Alpha. What's similar or different in how the 5's are used in your expression as compared to the other two expressions?

I added 5 to the number of blocks in the previous building while they used the building number.

Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

How does that change things?

Well, to find the cubes in Building 10 I'd have to work all the way up to 10.

Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

That's right, when you have an equation written with an "n" we just need to know the building number and plug it in, but with the $x + 5$, you have to know all the blocks of cubes in the building up to that point.

Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$ explicit
 $5n - 4$ explicit
 $x + 5$ recursive

When you need to know the number of cubes for the buildings up to the one you want to find, it is called a recursive formula, while an equation that requires only the input value is called an explicit formula.

Analyzing Lesson Sketch Episodes



Mathematical
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Scenario E

Study the sequence of cube buildings below.
Assuming the sequence continues in the same way,
how many cubes will there be in the 4th building?
The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$

Our group noticed that the number of cubes increased by 5 for each building, one on each arm, so we wrote the expression $x + 5$.

Study the sequence of cube buildings below.
Assuming the sequence continues in the same way,
how many cubes will there be in the 4th building?
The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

Interesting. It sounds like your group was thinking of what changes from building to building. Is that right?

Yes, that's exactly what we did.

Study the sequence of cube buildings below.
Assuming the sequence continues in the same way,
how many cubes will there be in the 4th building?
The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

So you used the plus 5 to show that you added 5 to the number of blocks in the previous building?

Yes, that's right.

So to find the number of cubes in Building 10 you'd have to work all the way up to 10.

Yeah, but now I'm not so sure our method is a good one.

Study the sequence of cube buildings below.
Assuming the sequence continues in the same way,
how many cubes will there be in the 4th building?
The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

Well it works, but you would have to figure out the answers for all the buildings up to the one you're trying to find. But when you have an equation written with an "n" you just need to know the building number and plug it in.

Oh. That sounds easier.

Study the sequence of cube buildings below.
Assuming the sequence continues in the same way,
how many cubes will there be in the 4th building?
The 17th building? The n th building.

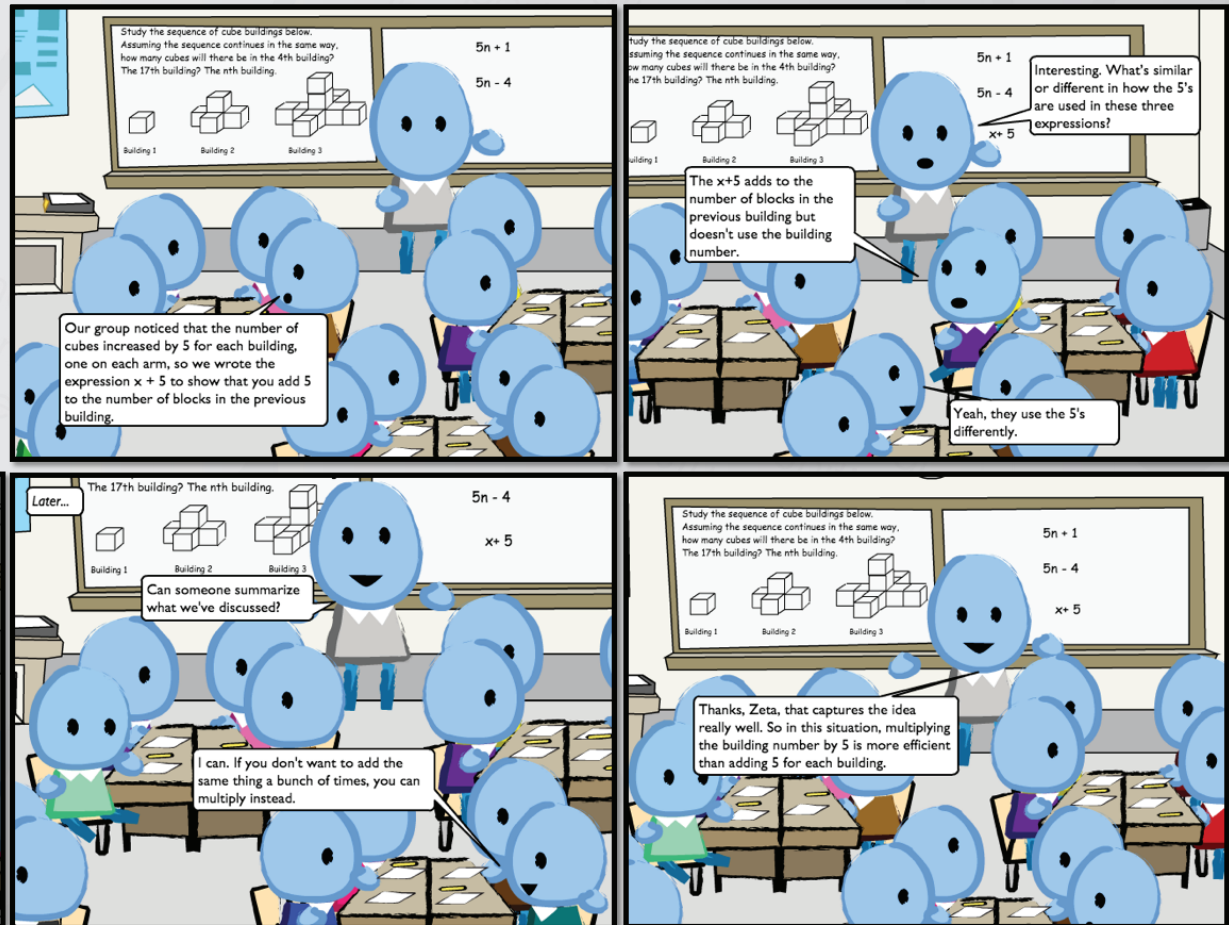
Building 1 Building 2 Building 3

$5n + 1$ explicit
 $5n - 4$ explicit
 $x + 5$ recursive

Okay class, let me give you some vocabulary to go with these ideas. When you need to know the number of cubes for the buildings up to the one you want to find, it is called a recursive formula, while an equation that requires only the input value is called an explicit formula.

Analyzing Lesson Sketch Episodes

Scenario F



Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$

Our group noticed that the number of cubes increased by 5 for each building, one on each arm, so we wrote the expression $x + 5$ to show that you add 5 to the number of blocks in the previous building.

Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

Interesting. What's similar or different in how the 5's are used in these three expressions?

The $x+5$ adds to the number of blocks in the previous building but doesn't use the building number.

Yeah, they use the 5's differently.

That's right; the $x+5$ adds the 5 to the number of blocks in the previous building and in the other equations the building number is multiplied by 5.

So what is the relationship between addition and multiplication?

Multiplication is a faster way of adding a bunch of things.

The discussion continues...

Later... The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n - 4$
 $x + 5$

Can someone summarize what we've discussed?

I can. If you don't want to add the same thing a bunch of times, you can multiply instead.

Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The n th building.

Building 1 Building 2 Building 3

$5n + 1$
 $5n - 4$
 $x + 5$

Thanks, Zeta, that captures the idea really well. So in this situation, multiplying the building number by 5 is more efficient than adding 5 for each building.

Building on Student Mathematical Thinking



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Make student thinking an object of consideration for the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.

Building on Student Mathematical Thinking



Mathematical
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*Make **student thinking** an object of consideration for the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.*

0. Invite/allow students to share their mathematical thinking

Building on Student Mathematical Thinking



*Make **student thinking** an object of consideration for the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.*

0. Invite/allow students to share their mathematical thinking (**elicit**)

Building on Student Mathematical Thinking



*Make **student thinking** an object of consideration for the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.*

0. Invite/allow students to share their mathematical thinking (**elicit**)
1. Make the object of consideration clear

Building on Student Mathematical Thinking



*Make **student thinking** an object of consideration for the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.*

0. Invite/allow students to share their mathematical thinking (**elicit**)
1. Make the object of consideration clear (**make precise**)

Building on Student Mathematical Thinking



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*Make student thinking **an object of consideration for the class** in order to engage the class in making sense of that thinking to better understand an important mathematical idea.*

0. Invite/allow students to share their mathematical thinking (**elicit**)
1. Make the object of consideration clear (**make precise**)
2. Turn the object of consideration over to the students

Building on Student Mathematical Thinking



Mathematical
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*Make student thinking **an object of consideration for the class** in order to engage the class in making sense of that thinking to better understand an important mathematical idea.*

0. Invite/allow students to share their mathematical thinking (**elicit**)
1. Make the object of consideration clear (**make precise**)
2. Turn the object of consideration over to the students (**grapple toss**)

Building on Student Mathematical Thinking



Mathematical
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*Make student thinking an object of consideration for the class in order to **engage the class in making sense of that thinking** to better understand an important mathematical idea.*

0. Invite/allow students to share their mathematical thinking (**elicit**)
1. Make the object of consideration clear (**make precise**)
2. Turn the object of consideration over to the students (**grapple toss**)
3. Orchestrate the students' process of making sense of the thinking

Building on Student Mathematical Thinking



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*Make student thinking an object of consideration for the class in order to **engage the class in making sense of that thinking** to better understand an important mathematical idea.*

0. Invite/allow students to share their mathematical thinking (**elicit**)
1. Make the object of consideration clear (**make precise**)
2. Turn the object of consideration over to the students (**grapple toss**)
3. Orchestrate the students' process of making sense of the thinking (**orchestrate**)

Building on Student Mathematical Thinking



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*Make student thinking an object of consideration for the class in order to engage the class in making sense of that thinking **to better understand an important mathematical idea.***

0. Invite/allow students to share their mathematical thinking (**elicit**)
1. Make the object of consideration clear (**make precise**)
2. Turn the object of consideration over to the students (**grapple toss**)
3. Orchestrate the students' process of making sense of the thinking (**orchestrate**)
4. Facilitate the extraction of the important mathematical idea

Building on Student Mathematical Thinking



Mathematical
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*Make student thinking an object of consideration for the class in order to engage the class in making sense of that thinking **to better understand an important mathematical idea.***

0. Invite/allow students to share their mathematical thinking (**elicit**)
1. Make the object of consideration clear (**make precise**)
2. Turn the object of consideration over to the students (**grapple toss**)
3. Orchestrate the students' process of making sense of the thinking (**orchestrate**)
4. Facilitate the extraction of the important mathematical idea (**make explicit**)

Building on Student Mathematical Thinking



Make student thinking an object of consideration for the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea.

0. Invite/allow students to share their mathematical thinking (**elicit**)
1. Make the object of consideration clear (**make precise**)
2. Turn the object of consideration over to the students (**grapple toss**)
3. Orchestrate the students' process of making sense of the thinking (**orchestrate**)
4. Facilitate the extraction of the important mathematical idea (**make explicit**)

Discussion Questions



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- To what extent does this conceptualization of building resonate with your experience?
- How do you see this conceptualization as being useful in your practice as a mathematics teacher educator?
- What is the value of unpacking mathematics teaching practices to this level?

Contact Information



Mathematical
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Thinking

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